Holographic anatomy of fuzzballs

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ABSTRACT: We present a comprehensive analysis of 2-charge fuzzball solutions, that is, horizon-free non-singular solutions of IIB supergravity characterized by a curve on R^4. We propose a precise map that relates any given curve to a specific superposition of R ground states of the D1-D5 system. To test this proposal we compute the holographic 1-point functions associated with these solutions, namely the conserved charges and the vacuum expectation values of chiral primary operators of the boundary theory, and find perfect agreement within the approximations used. In particular, all kinematical constraints are satisfied and the proposal is compatible with dynamical constraints although detailed quantitative tests would require going beyond the leading supergravity approximation. We also discuss which geometries may be dual to a given R ground state. We present the general asymptotic form that such solutions must have and present exact solutions which have such asymptotics and therefore pass all kinematical constraints. Dynamical constraints would again require going beyond the leading supergravity approximation.

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1. Introduction, summary of results and conclusions

An interesting proposal for the gravitational nature of black hole microstates has emerged over the last few years [1–3]; see also [4–6], and [7] for a review. According to this proposal there should exist a horizon free geometry associated with each black hole microstate.[1] These solutions should approach the original black hole geometry asymptotically and they should generically differ from each other up to the horizon scale; in a sense the interior of the horizon is replaced by a fuzzball. In this picture the black hole provides only the average statistical description; the true description is in terms of the horizon free microstate geometries.

Such a picture would resolve long standing puzzles associated with black hole physics such as the information loss paradox since the underlying physics of the black hole would not be conceptually different from that of a distant star, with the temperature and entropy

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[1]Note however that in general only a subset of the solutions will have small enough curvatures to be accurately described by supergravity.

being of a statistical origin. It is thus important to scrutinize the current evidence and further develop this proposal.

The fuzzball proposal requires the existence of exponential numbers of horizon free solutions. So the most basic question is whether one can find such a number of solutions with the required properties. A crucial issue here is what precisely are the "required properties". Furthermore, to utilize this proposal and to address questions such as how the Hawking temperature and other black hole properties emerge one would like to understand the precise relation between solutions and microstates.

A test case for the fuzzball proposal has been the 2-charge D1-D5 system. This is a 1/4 supersymmetric system and the "naive" black hole geometry has a near-horizon geometry of the form $AdS_3 \times S^3 \times M$, where M is either T^4 or $K3$. The naive geometry has a naked singularity but one expects that a horizon would emerge from α' corrections. At any rate, the description in terms of D-branes (at weak coupling) is well defined and one can obtain a statistical entropy in much the same way as for the 3 charge geometry which has a finite radius horizon. Indeed, the D1-D5 system can be mapped by dualities to a system of a fundamental string carrying momentum modes and the degeneracy of the system can be computed by standard methods. To be more specific, let us take $M = T^4$; then the degeneracy is the same as that of 8 bosonic and 8 fermionic oscillators at level $N = n_1 n_5$, where n_1 and n_5 are the number of D1 and D5 branes, respectively. The fuzzball proposal in this context is that there should exist an exponential number of horizon free solutions, one for each microstate, each carrying these two D-brane charges.

An exponential number of solutions was constructed by Lunin and Mathur in [1] and proposed to correspond to microstates. These were found by dualizing a subset of the FP solutions [8, 9], namely those that are associated with excitations of four bosonic oscillators. These provide enough solutions to account for a finite fraction of the entropy but one still needs an exponential number of solutions (associated with the additional four bosonic and eight fermionic oscillators in the example of T^4) to account for the total entropy. Such solutions, related to the odd cohomology of T^4 and the middle cohomology of the internal manifold have been discussed in [10] and [3], respectively, and we will complete this program in a forthcoming publication [11]. We thus indeed find that there are an appropriate number of solutions to account for all of the D1-D5 entropy.[2]

Do these solutions, however, have the right properties to be associated with D1-D5 microstates, and if yes, what is the precise relation? The aim of this paper is to address this question for the solutions corresponding to the universal sector of the T^4 and $K3$ compactifications.

As mentioned above the solutions of interest were obtained by dualizing FP solutions so let us briefly review these solutions and their relation to string perturbative states. A more detailed discussion will be given in section 2. The FP solutions (which are general chiral null models) involve the metric, B-field and the dilaton and are characterized by a null curve $F^I(x^+)$ with $I = 1, \ldots, 8$ in R^8. The solution describes the long range fields

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[2]Note however that this is a continuous family of supergravity solutions. To properly count them one needs to appropriately quantize them. Such a quantization has been discussed in [12], see also [13, 14] for a counting using supertubes.

sourced by a string wrapping one compact direction and having a transverse profile given by the null curve $F^I(x^+)$. The ADM conserved charges, i.e. the mass, momentum and angular momentum, associated with this solution are given precisely by the energy, momentum and angular momentum of the classical string that sources the solution.

On general grounds, one would expect that this classical string should be produced by a coherent state of string oscillators. Indeed, we show in section 2 that associated to a classical curve $F^I(x^+)$,

$$F^I(x^+) = \sum_{n>0} \frac{1}{\sqrt{n}} \left(\alpha_n^I e^{-in\left(\frac{x^+}{wR_9}\right)} + (\alpha_n^I)^* e^{in\left(\frac{x^+}{wR_9}\right)} \right), \tag{1.1}$$

where $x^+ = x^0 + x^9$, x^9 is the compact direction of radius R_9, w is the winding number and α_n^I are (complex) numerical coefficients, there is a coherent state $|F^I)$ of the first quantized string in an unconventional lightcone gauge with $x^+ = wR_9\sigma_+$, where σ_+ is a worldsheet lightcone coordinate, such that the expectation value of all conserved charges match the conserved charges associated with the solution. More precisely, let

$$X^I = \sum_{n>0} \frac{1}{\sqrt{n}} \left(\hat{a}_n^I e^{-in\sigma_+} + (\hat{a}_n^I)^\dagger e^{in\sigma_+} \right) \tag{1.2}$$

be the 8 transverse left moving coordinates with $\hat{a}_n^I$ the quantum oscillators normalized such that $[\hat{a}_n^I, (\hat{a}_m^J)^\dagger] = \delta^{IJ}\delta_{mn}$. The corresponding coherent state is given by

$$|F^I) = \prod_{n,I} |\alpha_n^I) \tag{1.3}$$

where $|\alpha_n^I)$ is a coherent state of the left-moving oscillator $\hat{a}_n^I$, i.e. it satisfies $\hat{a}_n^I|\alpha_n^I) = \alpha_n^I|\alpha_n^I)$, and the eigenvalues α_n^I are the coefficients appearing in (1.1). By construction

$$(F^I|X^I|F^I) = F^I \tag{1.4}$$

with root mean deviation of order $1/\sqrt{m}$, where $m \equiv (F^I|\hat{m}|F^I)$ the expectation value of the occupation operator[3] $\hat{m} = \sum \hat{a}_{-n}^I \hat{a}_n^I$. In other words, the expectation value is given by the classical string that sources the solution, and this is an accurate description as long as the excitation numbers are high. For low excitation numbers the state produced is fuzzy and the supergravity solution would require quantum corrections (as one would indeed expect). Note that the right-movers are in their ground state throughout this discussion.

Given winding w and momentum p_9 quantum numbers there are also corresponding Fock states

$$\prod (\hat{a}_{-n^I}^I)^{m_I}|0\rangle, \qquad N_L = \sum n^I m^I = -wp_9 \tag{1.5}$$

where N_L is the total left-moving excitation level (m_I are integers). It is sometimes stated in the literature that the solutions of [8, 9] represent these states. This cannot be exactly

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[3]Usually the occupation operator is called N but we reserve this letter for the level of the Fock states, $N = \sum n\hat{a}_{-n}^I \hat{a}_n^I$. Note also that after the duality to the D1-D5 system the occupation number becomes the eigenvalue of j_3 which is usually called m.

correct as the string coordinates have zero expectation on these states, so semiclassically they do not produce the required source. The statement is however approximately correct since these states strongly overlap with the corresponding coherent state for high excitation numbers. So in the regime where supergravity is valid the coherent state can be approximated by Fock states. Notice that one can organize the Fock states (1.5) into eigenstates of the angular momentum operator by using as building blocks linear combination of oscillators that themselves are eigenstates (e.g. $(\hat{a}^I_{-n} \pm i\hat{a}^{I+1}_{-n})$). The coherent states are however (infinite) superpositions of states with different angular momenta and are thus not eigenstates of the angular momentum operator.

We now return to the discussion of the dual D1-D5 system. The solutions of [1] were obtained by dualizing the FP solutions we just discussed but with a curve that is restricted to lie on R^4. The corresponding underlying states are now R ground states of the CFT associated with the D1-D5 system. This CFT is a deformation of a sigma model with target space the symmetric product of the compactification manifold X, $S^N(X)$ ($N = n_1 n_5$ and n_1, n_5 are the number of D1 and D5 branes). The R ground states can be obtained by spectral flow of the chiral primaries of the NS sector. Recall that the chiral primaries are associated with the cohomology of the internal space. For the discussion at hand only the universal cohomology is relevant and this leads (after spectral flow) to the following R ground states

$$\prod (\mathcal{O}^{R(\pm,\pm)}_{n_l})^{m_l}|0\rangle \qquad \sum n_l m_l = N = n_1 n_5 \,, \tag{1.6}$$

where n_l is the twist, m_l are integers and the superscripts denote (twice) the R-charges of the operator. Here the ground states are described in the language of the orbifold CFT; each ground state of the latter will map to a ground state of the deformed CFT. Notice that there is 1-1 correspondence between these states and the Fock states in (1.5). Namely one can map the operators $\mathcal{O}^{R(\pm,\pm)}$ to the harmonic oscillators,[4]

$$\hat{a}^{\pm 12}_n \quad \leftrightarrow \quad \mathcal{O}^{R(\mp,\mp)}_n, \qquad \hat{a}^{\pm 34}_n \quad \leftrightarrow \quad \mathcal{O}^{R(\pm,\mp)}_n. \tag{1.7}$$

where $\hat{a}^{\pm 12}_n \equiv (\hat{a}^1_n \pm i\hat{a}^2_n)/\sqrt{2}$ and $\hat{a}^{\pm 34}_n \equiv (\hat{a}^3_n \pm i\hat{a}^4_n)/\sqrt{2}$. In particular, the frequency n is mapped to the twist of the operator and the R-charge to the angular momentum in the 1-2 and 3-4 plane. However, the underlying algebra of these operators is different from the algebra of the harmonic oscillators.

Motivated by this correspondence it was proposed in [1] that each of the solutions obtained via dualities from the FP solution corresponds to a R ground state and via spectral flow to a chiral primary [2]. One of the original motivations for this work was to understand how such a map might work. Whilst it was clear from these works that the frequencies involved in the Fourier decomposition of the curve should map to twists of operators, it was unclear what the meaning of the amplitudes is in general and moreover a generic curve has far more parameters than an operator of the form (1.6). In our discussion of the FP system we have seen that the geometry is more properly viewed as dual to a

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[4]This correspondence straightforwardly extends to the general case where all R ground states are considered and all bosonic and fermionic oscillators are used in (1.5)

coherent state rather than a single Fock state. The coherent state however viewed as linear superposition of Fock states (see (2.18)) contains states that do not satisfy the constraint $N_L = -p_9 w$ and therefore do not map to R ground states after the dualities. This then leads to the following proposal for the map between geometries and states [15]:[5]

Given a curve $F^i(v)$ we construct the corresponding coherent state in the FP system and then find which Fock states in this coherent state satisfy $N_L = -p_9 w$. Applying the map (1.7) then yields the superposition of R ground states that is proposed to be dual to the D1-D5 geometry.

Let us see how this works in some simple examples. The simplest case is that of a circular planar curve that we may take to lie in the 1-2 plane:

$$F^1(v) = \frac{\sqrt{2N}}{n} \cos 2\pi n \frac{v}{L}, \qquad F^2(v) = \frac{\sqrt{2N}}{n} \sin 2\pi n \frac{v}{L}, \qquad F^3 = F^4 = 0, \qquad (1.8)$$

where L is the length of the curve and the overall factors are fixed by requiring that the solution has the correct charges (this will be explained in the main text). The corresponding coherent state can immediately be read off from the curve

$$|a_n^{-12}; a_n^{+12}; a_n^{-34}; a_n^{+34}) = |\sqrt{N/n}; 0; 0; 0).\qquad (1.9)$$

In this case there is a single state with $N_L = N = -wp_9$ contained in this coherent state, namely

$$|N/n\rangle = (\hat{a}_{-n}^{-12})^{N/n}|0\rangle.\qquad (1.10)$$

Using the map (1.7) we get that the D1-D5 solution based on the circle is dual to the R ground state

$$|circle) = \left(\mathcal{O}_n^{R(+,+)}\right)^{N/n}\qquad (1.11)$$

which was the proposal in [1].

As soon as one moves to more complicated curves, however, the correspondence becomes more complex, as there is more than one Fock state with $N_L = -wp_9$. For example the next simplest case is the solution based on an ellipse

$$F^1(v) = \frac{\sqrt{2N}}{n} a \cos 2\pi n \frac{v}{L}, \qquad F^2(v) = \frac{\sqrt{2N}}{n} b \sin 2\pi n \frac{v}{L}, \qquad F^3 = F^4 = 0, \qquad (1.12)$$

with $a^2 + b^2 = 2$. Following our prescription we obtain the following superposition

$$|ellipse) = \sum_{k=0}^{N/n} \frac{1}{2^{\frac{N}{n}}} \sqrt{\frac{(\frac{N}{n})!}{(\frac{N}{n}-k)!k!}} (a+b)^{\frac{N}{n}-k}(a-b)^k \left(\mathcal{O}_n^{R(+,+)}\right)^{\frac{N}{n}-k} \left(\mathcal{O}_n^{R(-,-)}\right)^k, \quad (1.13)$$

as is explained in section 2.3. The superposition for a general curve will involve a large number of Fock states.

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[5]A map between density matrices of the CFT states built from 4 bosonic oscillators and modified fuzzball solutions has been recently discussed in [16]. Here we provide a map between the original fuzzball solutions and superpositions of R ground states of the D1-D5 system.

Given such a map from curves to superpositions of states the question is whether the correspondence can be checked quantitatively. The D1-D5 solutions approach $AdS_3 \times S^3$ (times T^4 or $K3$) in the decoupling limit so one can use the AdS/CFT correspondence to make detailed quantitative tests. Recall that the deviations of the solution from $AdS_3 \times S^3$ encode vacuum expectation values of chiral primary operators (and possible deformations of the CFT by such operators), so by analyzing the asymptotics one can in principle completely characterize the ground state of the boundary theory.

Before proceeding to explain this, let us contrast the somewhat different meanings that one attaches to the statement "a geometry is dual to a state $|S\rangle$". In the context of the FP system, the state $|S\rangle$ is meant to provide the source for the supergravity solution and because of that we argued it should be a coherent state. In the context of the D1-D5 system however the same statement means that the ground state of the dual field theory is the state $|S\rangle$ (so $|S\rangle$ need not be approximated by a classical solution) and the vevs of gauge invariant operators on this state, $\langle S|\mathcal{O}|S\rangle$, are encoded in the asymptotics of the solution.

The D1-D5 system is governed by a 1+1 dimensional theory with $\mathcal{N} = (4, 4)$ supersymmetry. This theory has Coulomb and Higgs branches (which are distinct even quantum mechanically) [17 – 19]. The boundary CFT is the IR limit of the theory on the Higgs branch. Thus the fuzzball solutions should be in correspondence with the Higgs branch. Note that due to strong infrared fluctuations in 1+1 dimensions one usually encounters wavefunctions rather than continuous moduli spaces of the quantum states. So more properly one should view the fuzzball solutions as dual to wavefunctions on the Higgs branch. These wavefunctions, however, may be localized around specific regions in the large N limit and one should view our proposed correspondence in this way.

The vevs of gauge invariant operators in this 1+1 dimensional theory can be computed from the asymptotics of the solution. The existence of such a relationship follows from the basic AdS/CFT dictionary that relates bulk fields to boundary operators and the bulk partition function to boundary correlation functions. The implementation of this program is however very subtle and precise formulae for the 1-point functions for solutions with asymptotics to $AdS \times S$ were only recently obtained [20].

Naively the vev of an operator of dimension k is linearly related to coefficients of order z^k in the asymptotic expansion of the solution, where z is a radial coordinate (with the boundary of AdS located at $z = 0$.) The actual map however is more complicated and involves in addition a variety of non-linear contributions from terms of lower order z^l, $l < k$. There are four sources of such non-linear contributions, as we now discuss.

Recall that the holographic 1-point functions are derived by functionally differentiating the *renormalized* on-shell action w.r.t. the corresponding sources (see, for example, the review [21]). The most transparent way to describe the outcome of this computation is to use a radial Hamiltonian language where the radial coordinate plays the role of time. Ignoring for the moment the compact part of the geometry, so analyzing only the $(p+1)$ dimensional theory, one finds that the renormalized 1-point function of an operator of dimension k is exactly equal to the part of the corresponding radial canonical momentum that has dimension k [22, 23]. This coefficient is related to the asymptotic coefficients but the map is in

general non-linear (due to the non-linear nature of gravitational field equations). This is the first source of non-linearities and is essentially due to general covariance, since it is the canonical momentum that transforms properly under diffeomorphisms, not each coefficient of the asymptotic expansion. So although the relation between the 1-point function and momentum is linear, the relation with the asymptotic coefficients is non-linear [24].

Taking into account the compact part of the geometry leads to additional subtleties [20]. Firstly, one needs to understand the map between bulk fields and boundary operators beyond the linearized approximation, i.e. we need the non-linear Kaluza-Klein reduction map. Furthermore, this map should be gauge invariant i.e. independent of the parametrization of the compact space. The latter is dealt with by constructing gauge invariant variables, which are non-linear in terms of the original fields. Finally, 1-point functions may receive contributions from boundary terms in higher dimensions. Such contributions are responsible for the extremal correlators [25] and induce terms non-linear in momenta in the 1-point functions.

For the case at hand, the first step is to reduce the 10 dimensional solution over T^4 or $K3$. We show that the fuzzball solutions reduce to solutions of 6-dimensional supergravity coupled to tensor multiplets. These solutions (in the decoupling limit) are asymptotic to $AdS_3 \times S^3$. The next step is to find the non-linear gauge invariant KK map from 6 to 3 dimensions. Following [20], this is done to second order in the fluctuations using (and extending) the results of [26, 27]. The results up to this order are sufficient to derive (after taking into account the subtle issue of extremal correlators) the vevs of all 1/2 BPS operators up to dimension 2. This includes in particular the conserved charges and the stress energy tensor. We emphasize that the non-linear terms are crucial in getting the right physics. We also discuss the vevs of higher dimension operators but these results are only qualitative as we did not compute the non-linear contributions; these could be computed along the lines described above, but the computation becomes very tedious. One point functions for this system have also been discussed in the context of black rings [28], although the non-linear terms (which play a crucial role) were not included there.

The final results for the vevs of the fuzzball solution are given in section 6. In particular, the vevs of the stress energy is (non-trivially) zero for all solutions, consistent with the fact that the solutions are supersymmetric. The vevs of the other operators are

$$\left\langle \mathcal{O}_{S_i^1} \right\rangle = \frac{n_1 n_5}{4\pi}(-4\sqrt{2}f_{1i}^5); \qquad (i=1,\ldots,4) \tag{1.14}$$

$$\left\langle \mathcal{O}_{S_I^2} \right\rangle = \frac{n_1 n_5}{4\pi}(\sqrt{6}(f_{2I}^1 - f_{2I}^5)); \qquad (I=1,\ldots,9)$$

$$\left\langle \mathcal{O}_{\Sigma_I^2} \right\rangle = \frac{n_1 n_5}{4\pi}\sqrt{2}(-(f_{2I}^1 + f_{2I}^5) + 8a^{\alpha-}a^{\beta+}f_{I\alpha\beta}); \qquad (\alpha=1,\ldots,3)$$

$$\left\langle J^{+\alpha} \right\rangle = \frac{n_1 n_5}{2\pi}a^{\alpha+}(dy - dt); \qquad \left\langle J^{-\alpha} \right\rangle = -\frac{n_1 n_5}{2\pi}a^{\alpha-}(dy + dt),$$

where $\mathcal{O}_{S_i^1}$ are dimension 1 operators, $\mathcal{O}_{S_I^2}, \mathcal{O}_{\Sigma_I^2}$ are dimension 2 operators, and $J^{\pm\alpha}$ are R-symmetry currents. These operators correspond to the lowest lying KK states, the KK spectrum consisting of two towers of spin 1 supermultiplets, the S and Σ towers, and a tower of spin 2 supermultiplets, which contain the gauge field that is dual to the R-symmetry current. The coefficients $f_{1i}^5, f_{2I}^1, f_{2I}^5, a^{\pm\alpha}$ appear in the asymptotic expansion

of the harmonic functions that specify the solution, see (4.3)–(4.18), and $f_{I\alpha\beta}$ is a certain triple overlap of spherical harmonics. Expressed in terms of the defining curve F^i, the degree k coefficients involve symmetric rank k polynomials of F^i, see (4.6). In general, the vev of an operator of dimension k depends linearly on degree k coefficients and non-linearly on lower degree coefficients but such that the sum of degrees is k.

Any proposal for the field theory dual of these geometries should reproduce these vevs. Now, except when the curve is circular, operators charged wrt the R-symmetry acquire a vev. This implies immediately that the ground state of the field theory dual cannot be an eigenstate of R-symmetry since if this were the case only neutral operators would acquire a vev [15]. So none of the fuzzball solutions, except the circular ones, can correspond to a single R-ground state. Indeed, we have argued above (and in [15]) that these solutions should instead be dual to particular superpositions of R-ground states.

To test this proposal we discuss in some detail the case of the ellipse, comparing the vevs extracted from the supergravity solution with those implicit from the corresponding superposition of states in the field theory. We find complete matching for all kinematical properties of these vevs, thus demonstrating the consistency of our proposal. Moreover, the first dynamical test - matching of the R charges - is passed. To match the other vevs would require a knowledge of certain multiparticle three point functions at strong coupling, and is thus not currently possible. However, approximating the required three point functions using free harmonic oscillators leads to vevs which are in remarkable agreement with those extracted from the supergravity solution. This agreement suggests that certain three point functions in the dual CFT may be well approximated by free field computations, a result which in itself merits further investigation. Our proposal therefore passes all kinematical and all accessible dynamical tests, with other dynamical tests requiring going beyond the supergravity approximation.

Given that the original fuzzball solutions do not correspond to single R ground states, one may wonder whether there are other supergravity solutions that do correspond to a given R ground state. A necessary condition for this would be that the vevs of all charged operators are all zero, and this will only be the case if the solution preserves an $SO(2) \times SO(2)$ symmetry (among the original solutions only the circular one had this symmetry). We give the most general asymptotic supergravity solution consistent with these requirements. Different solutions with such asymptotics are parametrized by the vevs of the neutral operators, and to obtain these vevs one needs the complete solutions.

One way to produce solutions with an $SO(2) \times SO(2)$ symmetry is to take appropriate superpositions of the non-symmetric solutions. We discuss how to do such an averaging in general and we work out the details for the ellipse and for a curve that is a straight line followed by a semi-circle. This latter case yields the Aichelburg-Sexl metric namely the metric describing a massless particle moving along a greater circle on S^3 and sitting at the center of AdS_3. Solutions with the same $SO(2) \times SO(2)$ symmetry can also be produced using disconnected circular curves; one would expect that such solutions are related to Coulomb rather than Higgs branch physics.

We then discuss the relationship between such symmetric geometries and R ground states. We argue that the vevs for neutral operators in a particular ground state can

be related to three point functions at the conformal point. Thus with knowledge of the latter one can distinguish whether a given geometry corresponds to a particular R ground state. However, we find that implementing this procedure generically requires going beyond the leading supergravity approximation: one would need to know three point functions of multi particle operators, not captured by supergravity, as well as $1/N$ corrections. Thus we cannot currently determine which geometries are indeed dual to R ground states; indeed even the solutions based on disconnected curves (which should be Coulomb branch) could not be ruled out.

So what do our results imply for the fuzzball program? Firstly, our results support the overall picture; the fuzzball solutions can be in correspondence with the black hole microstates in a way that is compatible with the AdS/CFT correspondence and our computations provide the most stringent test to date. The detailed correspondence however is more complicated than anticipated. In particular a generic fuzzball solution corresponds to a superposition of many R ground states, and in general one would need to go beyond the leading supergravity to properly describe the system, even in this simplest 2-charge system. It has long been appreciated that most of the fuzzball solutions, despite being regular, have regions of high curvature so are at best extrapolations of the actual solutions describing the microstates. Here we see that even for solutions with low curvature everywhere, such as the ones based on large ellipses, one needs to go beyond the leading supergravity to test any proposed correspondence.

There has been a lot of interest in finding and analyzing fuzzball geometries in systems with more charges which have classical horizons [29] but a precise matching between these geometries and black hole microstates has not been established. Such a matching is clearly necessary, both to demonstrate that the correct geometries have been identified and to find for what fraction of the total entropy these account. A precise correspondence would also be important in understanding the quantization of the geometries and, most importantly of all, how the black hole properties emerge.

A key result of our work is that the vevs encoded by a given geometry give significant information about the field theory dual, and distinguish between geometries with the same charges (mass, angular momentum). In particular, dipole and higher multipole moments are related to the vevs of operators with dimension two or greater. Vevs determined by kinematics can by themselves rule out proposed correspondences, as shown in [15] and here, and vevs determined by dynamics are strong tests of a given proposal, when they can be computed on both sides. In particular, whilst our solutions based on disconnected curves pass all kinematical tests to correspond to R ground states on the Higgs branch, they should be ruled out by dynamical tests.

Previous work has often focused on computing two point functions and relating them to those in the dual field theory, and vice versa, see for example [30], but extracting vevs is much easier, since one needs only the geometry itself, rather than solving fluctuation equations in the geometry. Thus one can easily extract vevs from geometries with few symmetries, where the corresponding fluctuation equations are intractable. It hence seems worthwhile to explore whether the techniques developed here can give useful information in the context of other fuzzball geometries. One can analyze any fuzzball geometry which

has a throat region using AdS/CFT techniques, with the formalism developed here being directly applicable to three charge black strings in six dimensions. Black rings in six dimensions could also be explored using the same formalism; indeed the extracted data should uniquely identify the field theory dual.

The plan of the paper is as follows. In section 2 we discuss the relationship between solitonic string supergravity solutions and coherent states of the fundamental string. In section 3 we introduce the dual solutions in the D1-D5 system, and discuss the embedding of their decoupling limit into 6-dimensional supergravity. In section 4 we discuss the asymptotic expansion of these six dimensional solutions near the $AdS_3 \times S^3$ boundary. In section 5 we explain how the vevs of field theory operators can be extracted from these asymptotics. In section 6 we give the explicit values of these vevs for the fuzzball solutions in full generality, and in section 7 we specialize to the examples of solutions sourced by circular and elliptical curves. In section 8 we recall relevant features of the dual field theory, and discuss how the vevs can be related to three point functions at the conformal point. In section 9 we move on to the correspondence between fuzzball geometries and superpositions of chiral primaries, giving evidence for our proposed correspondence in terms of the matching of the vevs for the ellipsoidal case. In section 10 we discuss the asymptotics of a geometry dual to a single chiral primary, and give some examples of solutions which have such asymptotics. In section 11 we discuss the correspondence between symmetric geometries and chiral primaries, emphasizing that dynamical tests require going beyond the leading supergravity approximation. In section 12 we discuss how the asymptotically flat part of the geometry can be included in the field theory description.

Throughout the paper we use a number of technical results which are contained in appendices. Appendix A contains various properties of S^3 spherical harmonics whilst appendix B proves an addition theorem for harmonic functions on R^4. Appendix C discusses the perturbative expansion of six-dimensional field equations about the $AdS_3 \times S^3$ background. Appendix D discusses the supergravity computation of certain three point functions, whilst appendix E contains a derivation of the one point function for the energy momentum tensor in this system. Appendix F concerns the three point functions in the orbifold CFT; we argue that these differ from those computed in supergravity and that they are therefore not protected by any non-renormalization theorem.

2. FP system and perturbative states

We begin by discussing solitonic string supergravity solutions and their relation to perturbative string states. The FP solutions are characterized by a curve $F^I(x^+)$ describing the transverse displacement of the string. For later purposes only 4 transverse directions will be excited so the curve is confined to R^4 but for now we keep the discussion general. The supergravity solution describing an oscillating string is given by [8, 9]

$$ds^2 = H^{-1}(-dx^- dx^+ + K(dx^+)^2 - 2A_I dx^I dx^+) + dx_I dx_I$$

$$H = 1 + \frac{Q_f}{|\vec{x} - \vec{F}(x^+)|^6}, \quad K = \frac{Q_f |\dot{F}|^2}{|\vec{x} - \vec{F}(x^+)|^6}, \quad A_I = \frac{Q_f \dot{F}_I}{|\vec{x} - \vec{F}(x^+)|^6} \tag{2.1}$$

with suitable B field and dilaton. Here $x^\pm = x^0 \pm x^9$ are lightcone coordinates, $\vec{x}$ are 8 transverse coordinates and $x^9 \equiv x^9 + 2\pi R_9$. $\dot{F}_I$ denotes the derivative with respect to x^+. The fundamental string charge Q_f is proportional to the number of fundamental strings. The ADM mass and momentum along the compact direction are respectively [8, 9]

$$M = kQ_f(1 + |\dot{F}|_0^2); \quad P^9 = -kQ_f|\dot{F}|_0^2, \tag{2.2}$$

where the subscript denotes the zero mode and $k = 3\omega_7/2\kappa^2$ with ω_7 the volume of the S^7. The angular momenta in the transverse directions are similarly given by

$$J^{IJ} = kQ_f(F^J\dot{F}^I - F^I\dot{F}^J)_0. \tag{2.3}$$

As we will review below, these are exactly the conserved quantities of a string which wraps around the compact direction w times and whose transverse profile is given by F^I.

2.1 String quantization

To relate the supergravity solutions to perturbative string states, let us consider quantizing a string propagating in a flat background; we discuss this in some detail since the preferred gauge choice is a non standard light cone gauge. The relevant part of the worldsheet action is

$$S = \frac{1}{4\pi\alpha'}\int d^2\sigma(\partial_+ X^M \partial_- X_M + \cdots), \tag{2.4}$$

where the worldsheet metric is gauge fixed to $-g_{\tau\tau} = g_{\sigma\sigma} = 1$. Fermions will not play any role in the discussion here and will be suppressed. We will also set $\alpha' = 2$ to simplify formulae. Null worldsheet coordinates are introduced by setting $\sigma^\pm = (\tau \pm \sigma)$ and a lightcone gauge can be chosen for V such that

$$X^+ = (w^+\sigma^+ + \tilde{w}^+\sigma^-). \tag{2.5}$$

A similar choice of lightcone gauge for open strings has been discussed in [31]. The other fields are then expanded in harmonics as

$$X^- = x^- + (w^-\sigma^+ + \tilde{w}^-\sigma^-) + \sum_n \frac{1}{\sqrt{|n|}}(a_n^- e^{-in\sigma^+} + \tilde{a}_n^- e^{-in\sigma^-}); \tag{2.6}$$

$$X^I = x^I + p^I(\sigma^+ + \sigma^-) + \sum_n \frac{1}{\sqrt{|n|}}(a_n^I e^{-in\sigma^+} + \tilde{a}_n^I e^{-in\sigma^-}).$$

Reality of X^M demands that $a_{-n}^M = (a_n^M)^\dagger$. The Virasoro constraints are

$$T_{++} = \partial_+ X^M \partial_+ X_M = 0; \quad T_{--} = \partial_- X^M \partial_- X_M = 0. \tag{2.7}$$

At the classical level this enforces

$$(-w^+w^- + (p^I)^2)\delta_{m,0} + i\frac{m}{\sqrt{|m|}}(w^+ a_m^- - 2p^I a_m^I) + \sum_n \frac{n(n-m)}{\sqrt{|n(n-m)|}} a_n^I a_{m-n}^I = 0;$$

$$(-\tilde{w}^+\tilde{w}^- + (p^I)^2)\delta_{m,0} + i\frac{m}{\sqrt{|m|}}(\tilde{w}^+ \tilde{a}_m^- - 2p^I \tilde{a}_m^I) + \sum_n \frac{n(n-m)}{\sqrt{|n(n-m)|}} \tilde{a}_n^I \tilde{a}_{m-n}^I = 0,$$

thereby determining the non-dynamical field X^- in terms of the dynamical transverse fields X^I, as in standard lightcone gauge. The conserved momentum and winding charges are given by

$$P^M = \frac{1}{4\pi} \int_0^{2\pi} d\sigma (\partial_\tau X^M); \quad W^M = \frac{1}{2\pi} \int_0^{2\pi} d\sigma (\partial_\sigma X^M), \quad (2.8)$$

which take the values

$$P^M = \left(\frac{1}{4}(w^- + w^+ + \tilde{w}^- + \tilde{w}^+), \frac{1}{4}(w^+ - w^- + \tilde{w}^+ - \tilde{w}^-), p^I \right); \quad (2.9)$$

$$W^M = \left(\frac{1}{2}(w^- + w^+ - \tilde{w}^- - \tilde{w}^+), \frac{1}{2}(w^+ - w^- - \tilde{w}^+ + \tilde{w}^-), 0 \right).$$

In order for the string not to wind the time direction, one thus needs

$$W^0 = \frac{1}{2}(w^- + w^+ - \tilde{w}^- - \tilde{w}^+) = 0. \quad (2.10)$$

We are interested in states with only left moving excitations and no transverse momentum, namely the $\tilde{w}^+ = 0$ sector. For these the momentum and winding charges are

$$P^M = \left(\frac{1}{2}wR_9 - \frac{p_9}{R_9}, \frac{p_9}{R_9}, 0 \right); \quad W^M = (0, wR_9, 0); \quad (2.11)$$

$$w^+ \equiv wR_9; \quad w^- \equiv -\frac{2p_9}{R_9}.$$

Restricting to such states the L_0 constraint becomes

$$p_9 w + \sum_{n>0} n a_{-n}^I a_n^I \equiv p_9 w + N_L = 0. \quad (2.12)$$

The angular momenta in the transverse directions are given by the usual expressions

$$J^{IJ} = \frac{1}{4\pi} \int_0^{2\pi} d\sigma (X^J \partial_\tau X^I - X^I \partial_\tau X^J) = -i \sum_{n>0} (a_{-n}^I a_n^J - a_{-n}^J a_n^I). \quad (2.13)$$

Quantization proceeds in the standard way, with the oscillators satisfying the commutation relations

$$\left[\hat{a}_n^I, (\hat{a}_m^J)^\dagger \right] = \delta_{m,n} \delta^{IJ}, \quad (2.14)$$

and states being built out of creation operators $(\hat{a}_m^I)^\dagger$ acting on the vacuum. The classical expressions continue to hold, replacing a_m^I by operators $\hat{a}_m^I$, with appropriate shift in L_0 (which is negligible in the large charge limit).

2.2 Relation to classical curves

On rather general grounds, one expects that the supergravity solution characterized by a null curve corresponds to a coherent state of string oscillators. To be more precise, let us Fourier expand the classical curve

$$F^I(x^+) = \sum_{n>0} \frac{1}{\sqrt{n}} \left(\alpha_n^I e^{-in\sigma^+} + (\alpha_n^I)^* e^{in\sigma^+} \right) \quad (2.15)$$

where α_n^I are (complex) numerical coefficients and $x^+ = wR_9\sigma^+$. Then the coherent state $|F^I)$ of string oscillators that corresponds to this curve is given by

$$|F^I) = \prod_{n,I} |\alpha_n^I) \tag{2.16}$$

where $|\alpha_n^I)$ is a coherent state of the oscillator $\hat{a}_n^I$, i.e. it satisfies,

$$\hat{a}|\alpha) = \alpha|\alpha) \tag{2.17}$$

where we suppress the super and subscripts for clarity. Recall the coherent states are related to the Fock states by

$$|\alpha) = e^{-|\alpha|^2/2} \sum_k \frac{\alpha^k}{\sqrt{k!}}|k\rangle \tag{2.18}$$

and

$$|k\rangle = \frac{1}{\sqrt{k!}}(\hat{a}^\dagger)^k|0\rangle \tag{2.19}$$

is the standard kth excited state. By construction

$$(F^I|\hat{N}_L|F^I) \equiv N_L = \sum_{n>0} n|\alpha_n^I|^2. \tag{2.20}$$

From (2.12) and (2.11) we find that

$$(F^I|\hat{P}^0|F^I) = \left(\frac{1}{2}wR_9 + \frac{1}{wR_9}N_L\right); \quad (F^I|\hat{P}^9|F^I) = -\frac{1}{wR_9}N_L. \tag{2.21}$$

Now note that the zero mode of $(\dot{F}^I)^2$ is given by $2N_L/(wR_9)^2$. This means that the mass and momentum of the supergravity solution associated with this curve are, using (2.2),

$$M = kQ_f\left(1 + \frac{2N_L}{(wR_9)^2}\right); \quad P^9 = -kQ_f\frac{2N_L}{(wR_9)^2}, \tag{2.22}$$

which agree with the expressions (2.21) provided that

$$kQ_f = \frac{1}{2}wR_9, \tag{2.23}$$

which is the relationship found in [8, 9]. Moreover,

$$(F^I|\hat{J}^{IJ}|F^I) = \frac{1}{2}wR_9(F^J\dot{F}^I - F^I\dot{F}^J)_0, \tag{2.24}$$

which manifestly agrees with the expression (2.3).

2.3 Examples

Consider an elliptical curve in the 1-2 plane, such that

$$F^1 = \frac{\sqrt{2N}}{n} a \cos(n\sigma^+); \quad F^2 = \frac{\sqrt{2N}}{n} b \sin(n\sigma^+), \tag{2.25}$$

with $(a^2 + b^2) = 2$; this case was discussed in the introduction around (1.8) and (1.12). The amplitude of the curve is fixed such that the angular momentum in the 1-2 plane is

$$J^{12} = -\frac{N}{n} ab, \tag{2.26}$$

and the total excitation number defined in (2.20) is $N_L = N = -wp_9$. This ensures that the mass and momenta match that of the supergravity solution, as described in the previous subsection.

Introducing the usual combinations of oscillators with definite angular momenta in the 1-2 plane

$$\hat{a}_n^{\pm 12} \equiv \frac{1}{\sqrt{2}} (\hat{a}_n^1 \pm i\hat{a}_n^2), \tag{2.27}$$

the coherent state corresponding to the curve is

$$|a_n^{-12}; a_n^{+12}) = |\frac{\sqrt{N}}{2\sqrt{n}}(a+b); \frac{\sqrt{N}}{2\sqrt{n}}(a-b)), \tag{2.28}$$

which in the case of the circle ($\alpha = \beta$) reduces to (1.9). Extracting from this coherent state those states which satisfy $N_L = N$ gives

$$|ellipse) = \sum_{k=0}^{N/n} \frac{1}{2^{\frac{N}{n}}} \sqrt{\frac{(\frac{N}{n})!}{(\frac{N}{n}-k)!k!}} (a+b)^{\frac{N}{n}-k}(a-b)^k |k_{-12} = \left(\frac{N}{n} - k\right); k_{+12} = k\rangle, \tag{2.29}$$

which leads to the corresponding superposition (1.13) in the dual D1-D5 system.

3. The fuzzball solutions

We now consider the two charge fuzzball solutions in the D1-D5 system, obtained from the FP chiral null models by a chain of dualities. These fuzzball solutions were constructed by Lunin and Mathur [4, 1] and are given by

$$
\begin{aligned}
ds^2 &= f_1^{-1/2} f_5^{-1/2} \left(-(dt - A)^2 + (dy + B)^2 \right) + f_1^{1/2} f_5^{1/2} dx \cdot dx + f_1^{1/2} f_5^{-1/2} dz \cdot dz; \\
e^{2\Phi} &= f_1 f_5^{-1}; \\
C_{ti} &= f_1^{-1} B_i; \quad C_{ty} = f_1^{-1}; \\
C_{yi} &= f_1^{-1} A_i; \quad C_{ij} = c_{ij} - f_1^{-1}(A_i B_j - A_j B_i),
\end{aligned} \tag{3.1}
$$

where i, j are vector indices in the transverse R^4 and the metric is in the string frame. These fields solve the equations of motion following from the type IIB action

$$S = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-g_{10}} \left(e^{-2\Phi}(R_{10} + 4(\partial\Phi)^2) - \frac{1}{12} F_3^2 + \cdots \right), \tag{3.2}$$

where F_3 is the curvature of the two form C and $2\kappa_{10}^2 = (2\pi)^7 (\alpha')^4$ (we set $g_s = 1$ since it plays no role in our discussion), provided the following equations hold

$$dc = *_4 df_5, \quad dB = *_4 dA,$$
$$\Box_4 f_1 = \Box_4 f_5 = \Box_4 A_i = 0, \qquad \partial^i A_i = 0. \tag{3.3}$$

where the Hodge dual $*_4$ and $\Box_4$ are defined on the four (flat) non-compact overall transverse directions x^i. The compact part of the geometry does not play a role; it could be either T^4 or $K3$.

A solution to the conditions (3.3) based on an arbitrary closed curve $F^i(v)$ of length L in R^4 is given by

$$f_5 = 1 + \frac{Q_5}{L} \int_0^L \frac{dv}{|x - F|^2}; \quad f_1 = 1 + \frac{Q_5}{L} \int_0^L \frac{dv |\dot{F}|^2}{|x - F|^2}; \quad A_i = \frac{Q_5}{L} \int_0^L \frac{\dot{F}_i dv}{|x - F|^2}. \tag{3.4}$$

It was argued in [1] that these solutions are related to the R ground states (and via spectral flow to chiral primaries [2]) common to both the T^4 and $K3$ CFTs. The physical interpretation of these solutions is that the D1 and D5 brane sources are distributed on a curve in the transverse R^4. The D5-branes are uniformly distributed along this curve, but the D1-brane density at any point on the curve depends on the tangent to the curve. The total one brane charge is given by

$$Q_1 = \frac{Q_5}{L} \int_0^L |\dot{F}|^2 dv. \tag{3.5}$$

Both the Q_i have dimensions of length squared and are related to the integral charges by

$$Q_1 = \frac{(\alpha')^3 n_1}{V}; \quad Q_5 = \alpha' n_5, \tag{3.6}$$

where $(2\pi)^4 V$ is the volume of the compact manifold. Furthermore, the length of the curve is given by

$$L = 2\pi Q_5 / R, \tag{3.7}$$

where R is the radius of the y circle.

The holographic analysis in this paper will be done for the general class of solutions (3.1) satisfying (3.3). Results appropriate for the solutions determined by (3.4) will be obtained by specializing the general results to this case and we will indicate how this is done at each step of the analysis.

3.1 Compactification to six dimensions

Since only the breathing mode of the compact manifold is excited, it is convenient to compactify and work with solutions of six-dimensional supergravity. The effective six-dimensional (Einstein) metric coincides with the six-dimensional part of the (string frame) metric above (because the would be six-dimensional dilaton $\phi_6 = \Phi - \frac{1}{4} \ln \det g_{M^4}$, where g_{M^4} is the metric on the compact space, is constant). Thus the six-dimensional metric

$$ds^2 = f_1^{-1/2} f_5^{-1/2} \left(-(dt - A)^2 + (dy + B)^2 \right) + f_1^{1/2} f_5^{1/2} dx \cdot dx \tag{3.8}$$

along with the scalar field and tensor field of (3.1) satisfy the equations of motion following from the reduced action

$$S = \frac{1}{2\kappa_6^2} \int d^6x \sqrt{-g} \left(R - (\partial\Phi)^2 - \frac{1}{12}e^{2\Phi}F_3^2 \right), \tag{3.9}$$

where R is the six-dimensional curvature and F_3 is the curvature of the antisymmetric tensor field C. These equations of motion are

$$R_{MN} = \frac{1}{4}e^{2\Phi}\left(F_{MPQ}F_N{}^{PQ} - \frac{1}{6}F^2 g_{MN} \right) + \partial_M\Phi\partial_N\Phi;$$

$$D_M(e^{2\Phi}F^{MNP}) = 0; \quad \Box\Phi = \frac{1}{12}e^{2\Phi}F^2. \tag{3.10}$$

Note that the six-dimensional scalar field originates from the breathing mode of the compactification manifold.

The equations of motion which follow from the action (3.9) can be embedded into those of $d = 6$, $N = 4b$ supergravity coupled to n_t tensor multiplets, the covariant field equations for which were constructed in [32]. The bosonic field content of this theory is the graviton and five self-dual tensor fields from the supergravity multiplet, along with n_t anti-self dual tensor fields and $5n_t$ scalars from the tensor multiplets.

Following the notation of [33, 26] the bosonic field equations may be written as

$$R_{MN} = H^m_{MPQ}H^m_N{}^{PQ} + H^r_{MPQ}H^r_N{}^{PQ} + 2P^{mr}_M P^{mr}_N; \tag{3.11}$$

$$D^M P^{mr}_M = \frac{\sqrt{2}}{3}H^{mMNP}H^r_{MNP}, \tag{3.12}$$

along with Hodge duality conditions on the 3-forms

$$H^m_{MNP} = \frac{1}{6}\epsilon_{MNPQRS}H^{mQRS}; \quad H^r_{MNP} = -\frac{1}{6}\epsilon_{MNPQRS}H^{rQRS}. \tag{3.13}$$

In these equations m, n are SO(5) vector indices running from 1 to 5 whilst r, s are SO(n_t) vector indices running from 6 to $5 + n_t$. The three form field strengths are given by

$$H^m = G^A V^m_A; \quad H^r = G^A V^r_A, \tag{3.14}$$

where $A \equiv \{n, r\} = 1, \cdots, 5 + n_t$; $dG^A = 0$ and the vielbein on the coset space SO($5, n_t$)/(SO(5) $\times$ SO(n_t)) satisfies

$$V^m_A V^m_B - V^r_A V^r_B = \eta_{AB}, \tag{3.15}$$

with $\eta_{AB} = \text{diag}(+ + + + + - - - - \cdots -)$. The associated connection is

$$dVV^{-1} = \begin{pmatrix} Q^{mn} & \sqrt{2}P^{ms} \\ \sqrt{2}P^{nr} & Q^{rs} \end{pmatrix}. \tag{3.16}$$

The equations of motion (3.10) can be embedded into this theory using an SO(1, 1) subgroup as follows. Let

$$V_5^{m=5} = \cosh(\Phi); \quad V_6^{m=5} = \sinh(\Phi); \quad V_5^{r=6} = \sinh(\Phi); \quad V_6^{r=6} = \cosh(\Phi), \tag{3.17}$$

so that the connection is $\sqrt{2}P^{56} = d\Phi$. Now let[6]

$$G^5 = \frac{1}{4}(F + e^{2\Phi} *_6 F); \quad G^6 = \frac{1}{4}(F - e^{2\Phi} *_6 F), \tag{3.18}$$

which are both closed using the three form equation in (3.10). This implies that

$$H^{m=5} = \frac{1}{4}e^{\Phi}(F + *_6 F); \quad H^{r=6} = \frac{1}{4}e^{\Phi}(F - *_6 F), \tag{3.19}$$

which manifestly have the correct Hodge duality properties to satisfy (3.13). Substituting H and P into (3.11) also correctly reproduces the Einstein and scalar field equations of (3.10).

Since this embedding uses only an $SO(1,1)$ subgroup it does not depend on the details of the compactification manifold. Thus one can use this six-dimensional supergravity to analyze the fuzzball geometries in both T^4 and $K3$ systems. More generally, the (anomaly free) case of $n_t = 21$ gives the complete six dimensional theory obtained by K3 compactification of type IIB supergravity. For T^4 compactification of type IIB one obtains the maximally supersymmetric non-chiral six-dimensional theory, whose field content is a graviton, eight gravitinos, 5 self-dual and 5 anti-self dual three forms, 16 gauge fields, 40 fermions and 25 scalars. (Bosonic) solutions of this supergravity which do not have gauge fields switched on are solutions of the chiral supergravity given above, with $n_t = 5$.

3.2 Asymptotically AdS limit

In the appropriate decoupling limit, the solutions (3.1) become asymptotically AdS. This corresponds to harmonic functions with leading behavior r^{-2}. In terms of the harmonic functions in (3.4) the decoupling limit amounts to removing the constant terms in the harmonic functions f_1 and f_5. (Later on in section 12 we will discuss the interpretation of these constant terms in the dual CFT.) The solutions are then manifestly asymptotic to $AdS_3 \times S^3$ as $r \to \infty$. Firstly the metric asymptotes to

$$ds_6^2 = \frac{r^2}{\sqrt{Q_1 Q_5}}(-dt^2 + dy^2) + \sqrt{Q_1 Q_5}\left(\frac{dr^2}{r^2} + d\Omega_3^2\right); \tag{3.20}$$

whilst the three-forms and scalar field from (3.1) asymptote to

$$F_{rty} = \frac{2r}{Q_1}; \quad F_{\Omega_3} = 2Q_5; \quad e^{2\Phi_0} = \frac{Q_1}{Q_5}. \tag{3.21}$$

It is convenient to shift the scalar field so that $\Phi \to \Phi - \Phi_0$ and rescale $G^5 \to e^{\Phi_0}G^5$ and same for G^6. Then the relevant background fields of the six-dimensional supergravity are

$$g^{o(m=5)} = H^{o(m=5)} = \frac{r}{\sqrt{Q_1 Q_5}}dr \wedge dt \wedge dy + \sqrt{Q_1 Q_5}d\Omega_3; \tag{3.22}$$

$$V_5^{o(m=5)} = 1; \quad V_6^{o(r=6)} = 1,$$

with the off-diagonal components of the vielbein vanishing; the anti-self dual field $g^{o(r=6)} = H^{r=6}$ vanishing and Φ being zero also. Note that with the coordinate rescalings $t \to$

---

[6]The field strengths G^5 and G^6 were called $G^{\pm}$ in [15].

$t\sqrt{Q_1 Q_5}$ and $y \to y\sqrt{Q_1 Q_5}$, the curvature radius appears only as an overall scaling factor in both the metric (3.20) and the three form (3.22). When one rescales the coordinates in this way, the new y coordinate will have periodicity $\tilde{R} = R/\sqrt{Q_1 Q_5}$.

The goal is to extract from the subleading asymptotics around the AdS boundary the vevs of chiral primaries in the dual theory, and thus investigate the matching with R vacua. The strategy is as follows. First one expands the solution systematically near the AdS boundary. Then one extracts from the asymptotic solution the values of 6-dimensional gauge invariant fields. These must then be reduced to three dimensional fields using the KK map, and then the vevs can be extracted using holographic renormalization.

4. Harmonic expansion of fluctuations

Let us consider the asymptotic expansion of the solution. The perturbations of the six-dimensional supergravity fields relative to the $AdS_3 \times S^3$ background can be expressed as

$$g_{MN} = g_{MN}^o + h_{MN}; \quad G^A = g^{oA} + g^A; \quad \phi^{mr}. \tag{4.1}$$

These fluctuations can then be expanded in spherical harmonics as follows:

$$
\begin{aligned}
h_{\mu\nu} &= \sum h_{\mu\nu}^I(x) Y^I(y), \tag{4.2}\\
h_{\mu a} &= \sum (h_\mu^{I_v}(x) Y_a^{I_v}(y) + h_{(s)\mu}^I(x) D_a Y^I(y)),\\
h_{(ab)} &= \sum (\rho^{I_t}(x) Y_{(ab)}^{I_t}(y) + \rho_{(v)}^{I_v}(x) D_a Y_b^{I_v}(y) + \rho_{(s)}^I(x) D_{(a} D_{b)} Y^I(y)),\\
h_a^a &= \sum \pi^I(x) Y^I(y),\\
g_{\mu\nu\rho}^A &= \sum 3 D_{[\mu} b_{\nu\rho]}^{(A)I}(x) Y^I(y),\\
g_{\mu\nu a}^A &= \sum (b_{\mu\nu}^{(A)I}(x) D_a Y^I(y) + 2 D_{[\mu} Z_{\nu]}^{(A)I_v}(x) Y_a^{I_v}(y));\\
g_{\mu ab}^A &= \sum (D_\mu U^{(A)I}(x) \epsilon_{abc} D^c Y^I(y) + 2 Z_\mu^{(A)I_v} D_{[b} Y_{a]}^{I_v});\\
g_{abc}^A &= \sum (-\epsilon_{abc} \Lambda^I U^{(A)I}(x) Y^I(y));\\
\phi^{mr} &= \sum \phi^{(mr)I}(x) Y^I(y),
\end{aligned}
$$

Here (μ, ν) are AdS indices and (a, b) are S^3 indices, with x denoting AdS coordinates and y denoting sphere coordinates. The subscript (ab) denotes symmetrization of indices a and b with the trace removed. Relevant properties of the spherical harmonics are reviewed in appendix A. We will often use a notation where we replace the index I by the degree of the harmonic k or by a pair of indices (k, I) where k is the degree of the harmonic and I now parametrizes their degeneracy, and similarly for I_v, I_t.

Imposing the de Donder gauge condition $D^A h_{aM} = 0$ on the metric fluctuations removes the fields with subscripts (s, v). In deriving the spectrum and computing correlation functions, this is therefore a convenient choice. The de Donder gauge choice is however not always a convenient choice for the asymptotic expansion of solutions; indeed the natural coordinate choice in our application takes us outside de Donder gauge. As discussed in [20]

this issue is straightforwardly dealt with by working with gauge invariant combinations of the fluctuations; we will present the relevant gauge invariant combinations later.

4.1 Asymptotic expansion of the fuzzball solutions

Now consider the asymptotic expansion at large radius of the fuzzball solutions. The natural radial coordinate in which to expand the solutions is the radial coordinate r of the transverse R^4, even though with this choice it will turn out that the metric is not in de Donder gauge.

The harmonic functions appearing in the solution (3.1) can be expanded as

$$f_5 = \frac{Q_5}{r^2} \sum_{k,I} \frac{f_{kI}^5 Y_k^I(\theta_3)}{r^k};$$

$$f_1 = \frac{Q_1}{r^2} \sum_{k,I} \frac{f_{kI}^1 Y_k^I(\theta_3)}{r^k}; \tag{4.3}$$

$$A_i = \frac{Q_5}{r^2} \sum_{k,I} \frac{(A_{kI})_i Y_k^I(\theta_3)}{r^k},$$

for some coefficients f_{kI}^5, f_{kI}^1 and $(A_{kI})_i$. There are restrictions on the coefficients $(A_{kI})_i$ because $\partial^i A_i = 0$ which will be given below.

In the case of the (near-horizon) harmonic functions of (3.4), the coefficients $f_{kI}^5, f_{kI}^1, (A_{kI})_i$ are given in terms of the curve $F^i(v)$. To obtain these coefficients we make use of the following addition theorem for harmonic functions on R^4:

$$\frac{1}{(x^i - y^i)^2} = \sum_{k \geq 0} \frac{y^k}{(k+1)r^{2+k}} Y_k^I(\theta_3^x) Y_k^I(\theta_3^y). \tag{4.4}$$

In this expression x^i and y^i are Cartesian coordinates on R^4, with the corresponding polar coordinates being (r, θ_3^x) and (y, θ_3^y) respectively. $Y_k^I(\theta_3)$ are (normalized) spherical harmonics of degree k on S^3 with I labeling their degeneracy; the degeneracy of degree k harmonics is $(k+1)^2$. For the $k = 1$ harmonics of degeneracy four, it is convenient to use the label i, Y_1^i. The addition theorem can also be expressed as

$$\frac{1}{|x-y|^2} = \sum_{k \geq 0} \frac{1}{(k+1)r^{2+k}} Y_k^I(\theta_3^x)(C_{i_1 \cdots i_k}^I y^{i_1} \cdots y^{i_k}), \tag{4.5}$$

where $C_{i_1 \cdots i_k}^I$ are the orthogonal symmetric traceless rank k tensors on R^4 which are in one-to-one correspondence with the (normalized) spherical harmonics $Y_k^I(\theta_3)$ of degree k on the S^3. This formula is the exact analogue of the well-known addition theorem for electromagnetism (see [34]) and also of the addition theorem for harmonic functions on R^6 discussed in the appendix of [35], and it can be proved in the same way, as we show in appendix B.

Using the addition theorem we obtain

$$f_{kI}^5 = \frac{1}{(k+1)L} \int_0^L dv\, C_{i_1 \cdots i_k}^I F^{i_1} \cdots F^{i_k};$$

$$f_{kI}^1 = \frac{Q_5}{Q_1(k+1)L} \int_0^L dv\, \left|\dot{F}\right|^2 C_{i_1 \cdots i_k}^I F^{i_1} \cdots F^{i_k}; \qquad (4.6)$$

$$(A_{kI})_i = \frac{1}{(k+1)L} \int_0^L dv\, \dot{F}_i C_{i_1 \cdots i_k}^I F^{i_1} \cdots F^{i_k}.$$

Furthermore, in the final equality of (4.3) the summation is restricted to $k \geq 1$ because of the closure of the curve F^i ($\int dv \dot{F}_i = 0$). Note that we will often suppress implicit summations over the index I in later expressions for compactness.

Before substituting these expressions into the supergravity fields, we need to consider which fluctuations are physical. Suppose we use translational invariance to impose the condition

$$\int_0^L dv\, F_i = 0, \qquad (4.7)$$

which was the choice made in previous literature, for example, in [1]. This corresponds to choosing the origin of the coordinate system to be at the center of mass of the D5-branes. However, the center of mass of the D1-branes does not coincide with that of the D5-branes in general; thus this condition does *not* take one to the center of mass of the whole system.

Indeed with this choice the leading correction to the AdS background derives from the $k = 1$ terms in the harmonic function f_1. The choice (4.7) gives a leading metric deviation

$$h_{\mu\nu} = D_\mu D_\nu \lambda; \qquad h_{ab} = g_{ab}\lambda, \qquad (4.8)$$

with

$$\lambda = \sum_i \frac{f_{1i}^1 Y_1^i}{2r}, \qquad (4.9)$$

which satisfies $\Box\lambda = -\lambda$. Such a perturbation is unphysical because it can be removed by a superconformal transformation (with parameter $-\lambda$). The physical origin of the term is that with the choice (4.7) we are not working in the centre of mass of the system. Instead of imposing that the $k = 1$ term in the D5-brane harmonic function vanishes, we should impose that the $k = 1$ term in $\sqrt{f_1 f_5}$ vanishes, namely

$$f_{1i}^5 + f_{1i}^1 = 0. \qquad (4.10)$$

When the solution is related to a closed curve this reduces to

$$\int_0^L dv\, F^i \left(1 + \frac{Q_5}{Q_1}|\dot{F}|^2\right) = 0. \qquad (4.11)$$

Then all unphysical $k = 1$ terms in the metric vanish automatically.

Now consider the asymptotic expansion of A_i. The restriction on the coefficients in the asymptotic expansion imposed by the condition $\partial_i A^i = 0$ is most easily understood as follows. The form A may be written as

$$A = Q_5 \sum_{k,I,i} \frac{(A_{kI})_i}{r^{2+k}} Y_k^I \left(Y_1^i dr + r dY_1^i\right), \qquad (4.12)$$

using

$$dx^i = drY_1^i + rdY_1^i. \tag{4.13}$$

Projecting the products of spherical harmonics onto the basis of spherical harmonics gives

$$A = Q_5 \sum_{l,L,k,I,i} \frac{(A_{kI})_i}{r^{2+k}} \left(a_{iIL}Y_l^L dr + \frac{b_{IiL}}{\Lambda^L} rdY_l^L \right) \tag{4.14}$$

$$+ \quad Q_5 \sum_{k_v,I_v,k,I,i} \frac{(A_{kI})_i}{r^{1+k}} E_{I_vIi}^{\pm} Y_{k_v}^{I_v\pm},$$

where the spherical harmonic overlaps $(a_{iIJ}, b_{IiJ}, E_{I_vIi}^{\pm})$ are defined in (A.7), (A.6) and (A.10) respectively. The term in A proportional to the vector harmonic is already divergenceless on its own. The first two combine into divergenceless combination iff scalar harmonics with degree $l = (k-1)$ appear in this asymptotic expansion:

$$A = Q_5 \sum_{L,k,I,i} \frac{(A_{kI})_i}{r^{2+k}} a_{IiL} \left(Y_{k-1}^L dr - \frac{r}{(1+k)} dY_{k-1}^L \right) \tag{4.15}$$

$$+ Q_5 \sum_{k_v,I_v,k,I,i} \frac{(A_{kI})_i}{r^{1+k}} E_{I_vIi}^{\pm} Y_{k_v}^{I_v\pm}.$$

Vanishing of the other terms requires

$$\sum_{I,i} (A_{kI})_i a_{iIL} = 0 \qquad l \neq (k-1). \tag{4.16}$$

In particular this means that $(A_{1j})_i$ must be antisymmetric (since a_{ijL} is symmetric in i, j). Note that this condition is clearly satisfied for the $(A_{1j})_i$ defined in (4.6).

The leading term in the asymptotic expansion is given in terms of degree one vector harmonics as

$$A = \frac{Q_5}{r^2} (A_{1j})_i Y_1^j dY_1^i \equiv \frac{\sqrt{Q_5 Q_1}}{r^2} (a^{\alpha-} Y_1^{\alpha-} + a^{\alpha+} Y_1^{\alpha+}), \tag{4.17}$$

where $(Y_1^{\alpha-}, Y_1^{\alpha+})$ with $\alpha = 1, 2, 3$ form a basis for the $k = 1$ vector harmonics, which coincide with the Killing one forms of $SU(2)_L$ and $SU(2)_R$ respectively. Here we define

$$a^{\alpha\pm} = \frac{\sqrt{Q_5}}{\sqrt{Q_1}} \sum_{i>j} e_{\alpha ij}^{\pm} (A_{1j})_i \tag{4.18}$$

where the spherical harmonic triple overlap $e_{\alpha ij}^{\pm}$ is defined in (A.8) and explicit values in a particular basis are given in (A.22). For solutions defined by a curve $F^i(v)$, the coefficients $(A_{1j})_i$ are given in (4.6). The dual field to leading order is

$$B = \frac{\sqrt{Q_5 Q_1}}{r^2} (a^{\alpha-} Y_1^{\alpha-} - a^{\alpha+} Y_1^{\alpha+}), \tag{4.19}$$

where we use the Hodge duality property of the vector harmonics given in (A.2).

Putting these results together the leading perturbations of the metric are

$$-h_{tt} = h_{yy} = \frac{1}{2}\left(-(f^1_{2I} + f^5_{2I})Y^I_2 + (f^5_{1i}Y^i_1)^2\right);$$

$$h_{rr} = \frac{1}{2r^4}\left((f^1_{2I} + f^5_{2I})Y^I_2 - (f^5_{1i}Y^i_1)^2\right);$$

$$h_{ta} = \left(a^{\alpha-}Y^{\alpha-}_1 + a^{\alpha+}Y^{\alpha+}_1\right);$$

$$h_{ya} = \left(a^{\alpha-}Y^{\alpha-}_1 - a^{\alpha+}Y^{\alpha+}_1\right);$$

$$h_{ab} = g^o_{ab}\frac{1}{2r^2}\left((f^1_{2I} + f^5_{2I})Y^I_2 - (f^5_{1i}Y^i_1)^2\right) - \frac{2}{r^2}a^{\alpha-}a^{\beta+}((Y^{\alpha-}_1)_a(Y^{\beta+}_1)_b + (Y^{\alpha-}_1)_b(Y^{\beta+}_1)_a).$$

(4.20)

Note that the condition (4.10) has been used to eliminate f^1_{1i}. Terms quadratic in spherical harmonics will need to be projected back onto the basis of spherical harmonics in order to determine the contributions to each perturbation component in (4.2).

In these expressions we have suppressed the scale factor $\sqrt{Q_1Q_5}$. As mentioned previously, after rescaling $t \to t\sqrt{Q_1Q_5}$ and $y \to y\sqrt{Q_1Q_5}$, the metric has an overall scale factor $\sqrt{Q_1Q_5}$. Scale factors will similarly be suppressed in the other fields. The overall scaling will be taken into account via the normalization of the three-dimensional action.

Now consider the other supergravity fields; from (3.22) and (4.1) one finds the following three form fluctuations are

$$g^5_{tya} = \frac{1}{4}D_a\left(2(f^5_{1i}Y^i_1)^2 - (f^5_{2I} + f^1_{2I})Y^I_2\right);$$

$$g^5_{tab} = -(a^{\alpha-}D_{[a}(Y^{\alpha-}_1)_{b]} - a^{\alpha+}D_{[a}(Y^{\alpha+}_1)_{b]});$$

$$g^5_{yab} = -(a^{\alpha-}D_{[a}(Y^{\alpha-}_1)_{b]} + a^{\alpha+}D_{[a}(Y^{\alpha+}_1)_{b]});$$

(4.21)

$$g^5_{rab} = \frac{1}{r^3}\left(\frac{1}{4}\epsilon_{ab}{}^c(f^1_{2I} + f^5_{2I})D_cY^I_2 + 4a^{\alpha-}a^{\beta+}(Y^{\alpha-}_1)_{[a}(Y^{\beta+}_1)_{b]}\right);$$

$$g^5_{abc} = \frac{1}{r^2}\epsilon_{abc}(f^1_{2I} + f^5_{2I})Y^I_2 - \frac{6}{r^2}D_{[a}a^{\alpha-}a^{\beta+}(Y^{\alpha-}_1)_b(Y^{\beta+}_1)_{c]}).$$

and

$$g^6_{tyr} = \frac{1}{2}f^5_{1i}Y^i_1;$$

(4.22)

$$g^6_{tya} = \frac{1}{4}D_a\left(2f^5_{1i}Y^i_1r + (f^5_{2I} - f^1_{2I})Y^I_2\right);$$

$$g^6_{rab} = \frac{1}{2r^2}\epsilon_{ab}{}^c f^5_{1i}D_cY^i_1 + \frac{1}{4r^3}\epsilon_{ab}{}^c(f^5_{2I} - f^1_{2I})D_cY^I_2;$$

$$g^6_{abc} = \frac{3}{2r}\epsilon_{abc}f^5_{1i}Y^i_1 + \frac{1}{r^2}\epsilon_{abc}(f^5_{2I} - f^1_{2I})Y^I_2.$$

Finally the scalar field is expanded as

$$\phi^{(56)} \equiv \Phi = -\frac{f^5_{1i}}{r}Y^i_1 + \frac{1}{2}\frac{f^1_{2I} - f^5_{2I}}{r^2}Y^I_2.$$

(4.23)

All other fluctuations, g^A with $A \neq 5, 6$ and ϕ^{mr} with $m \neq 5$, $r \neq 6$ vanish.

4.2 Gauge invariant fluctuations

We now wish to extract gauge invariant combinations of these fluctuations. Gauge invariant means that the fluctuations do not transform under coordinate transformations $\delta x^M = \xi^M$, or, in the case of the three dimensional metric and gauge fields, they have the correct transformation properties. Using the fact that the metric and three forms transform (up to linear order in fluctuations) as

$$\delta h_{MN} = D_M \xi_N + D_N \xi_M + D_M \xi^P h_{PN} + D_N \xi^P h_{PM} - \xi^P D_P h_{MN}; \qquad (4.24)$$
$$\delta g^A_{MNP} = 3 D_{[M} \xi^S g^{oA}_{NP]S} + 3 D_{[M} \xi^S g^A_{NP]S} + \xi^S D_S g^A_{MNP},$$

one can systematically compute combinations which are gauge invariant to quadratic order in fluctuations. That is, the gauge invariant fluctuations $\hat{\psi}^Q$ are given by the following schematic expression

$$\hat{\psi}^Q = \sum_R a_{QR} \psi^R + \sum_{R,S} a_{QRS} \psi^R \psi^S, \qquad (4.25)$$

where ψ^Q collectively denotes all fields and the quadratic contributions are rather complicated in general. Note that each gauge invariant field at linearized level should reduce to the corresponding field in de Donder gauge on setting the fields with subscripts (s, v) to zero in (4.2). Clearly by retaining higher order terms in (4.24) one could compute the invariants to arbitrarily high order in the fluctuations.

For the discussion at hand, however, we do not need the most general expressions. Since we are working perturbatively in the radial coordinate, we need only retain terms in (4.25) with the same radial behavior. In particular, as we discuss only leading order and next to leading order perturbations, we will need at most quadratic invariants. In fact the only combinations which will be needed here are

$$\hat{\pi_2}^I = \pi_2^I + \Lambda^2 \rho^I_{2(s)}; \qquad (4.26)$$
$$\hat{U}_2^{(5)I} = U_2^{(5)I} - \frac{1}{2} \rho^I_{2(s)};$$
$$\hat{h}^0_{\mu\nu} = h^0_{\mu\nu} - \sum_{\alpha,\pm} h^{1\pm\alpha}_\mu h^{1\pm\alpha}_\nu.$$

In addition the fluctuations $(\Phi_1^i, \Phi_2^I, U_1^{(6)i}, U_2^{(6)I})$ are by themselves gauge invariant up to the necessary order and the fields $(Z_\mu^{(5)1\pm\alpha}, Z_\mu^{(6)1\pm\alpha}, h_\mu^{1\pm\alpha})$ by themselves transform correctly as gauge fields. Thus only in the metric do we need to take into account a quadratic contribution.

5. Extracting the vevs systematically

In this section we will compute the vevs following the systematic procedure of [20]. First one should identify the six-dimensional equations of motion that these fields satisfy to appropriate order, in this case quadratic. Secondly one should remove derivative terms in these equations of motion by a field redefinition: this defines the Kaluza-Klein reduction

map between six-dimensional and three-dimensional fields. Finally, once one has the three dimensional fields and their equations of motion, one extracts vevs using the by now familiar methods of holographic renormalization.

5.1 Linearized field equations

Let us first consider the linearized field equations. As discussed in [20], the equations of motion for the gauge invariant fields at linear order are precisely the same as those in de Donder gauge, provided one replaces all fields with the corresponding gauge invariant field. So now let us briefly review the linearized spectrum in de Donder gauge derived in [33]. Consider first the scalars. It is useful to introduce the following combinations of these fields which diagonalize the linearized equations of motion:

$$
\begin{aligned}
s_I^{(r)k} &= \frac{1}{4(k+1)}(\phi_I^{(5r)k} + 2(k+2)U_I^{(r)k}), \\
t_I^{(r)k} &= \frac{1}{4}(\phi_I^{(5r)k} - 2kU_I^{(r)k}), \\
\sigma_I^k &= \frac{1}{12(k+1)}(6(k+2)\hat{U}_I^{(5)k} - \hat{\pi}_I^k), \\
\tau_I^k &= \frac{1}{12(k+1)}(\hat{\pi}_I^k + 6k\hat{U}_I^{(5)k}).
\end{aligned}
\tag{5.1}
$$

Note that these combinations are applicable when the background $AdS_3 \times S^3$ has unit radius. Here the fields $s^{(r)k}$ and σ^k correspond to scalar chiral primaries. In what follows we will need only the $r = 6$ fields and will thus drop the r superscript. The masses of the scalar fields are

$$
m_{s^k}^2 = m_{\sigma^k}^2 = k(k-2), \quad m_{t^k}^2 = m_{\tau^k}^2 = (k+2)(k+4), \quad m_{\rho^k}^2 = k(k+2). \tag{5.2}
$$

Note also that $k \geq 0$ for $(\tau^k, t^{(r)k})$; $k \geq 1$ for $s^{(r)k}$; $k \geq 2$ for (σ^k, ρ^k).

Next consider the vector fields. It is useful to introduce the following combinations which diagonalize the equations of motion:

$$
h_{\mu I_v}^\pm = \frac{1}{2}(C_{\mu I_v}^\pm - A_{\mu I_v}^\pm), \quad Z_{\mu I_v}^{(5)\pm} = \pm\frac{1}{4}(C_{\mu I_v}^\pm + A_{\mu I_v}^\pm). \tag{5.3}
$$

For general k the equations of motion are Proca-Chern-Simons equations which couple $(A_\mu^\pm, C_\mu^\pm)$ via a first order constraint [33]. The three dynamical fields at each degree k have masses $(k-1, k+1, k+3)$, corresponding to dual operators of dimensions $(k, k+2, k+4)$ respectively. The lowest dimension operators are the R symmetry currents, which couple to the $k = 1$ $A_\mu^{\pm\alpha}$ bulk fields. The latter satisfy the Chern-Simons equation

$$
F_{\mu\nu}(A^{\pm\alpha}) = 0, \tag{5.4}
$$

where $F_{\mu\nu}(A^{\pm\alpha})$ is the curvature of the connection and the index $\alpha = 1, 2, 3$ is an SU(2) adjoint index. Only these bulk vector fields will be needed in what follows, and therefore the equations of motion for general k discussed in [33] are not given here. There are also

the massive vectors $Z_{\mu I_v}^{(6)\pm}$ but their mass is sufficiently high that they are irrelevant for our discussion.

Finally there is a tower of KK gravitons with $m^2 = k(k+2)$ but again only the massless graviton will play a role here. Note that it is the combination $\hat{H}_{\mu\nu} = h_{\mu\nu}^o + \pi^0 g_{\mu\nu}^o$ which satisfies the linearized massless Einstein equation

$$(\mathcal{L}_\mathcal{E} + 2)\hat{H}_{\mu\nu} \equiv \frac{1}{2}(-\Box \hat{H}_{\mu\nu} + D^\rho D_\mu \hat{H}_{\rho\nu} + D^\rho D_\nu \hat{H}_{\rho\mu} - D_\mu D_\nu \hat{H}_\rho^\rho + 4\hat{H}_{\mu\nu}) = 0. \quad (5.5)$$

That this is the appropriate combination follows from the reduction of the six-dimensional Einstein term in the action over the sphere; keeping terms linear in fluctuations the three dimensional action is

$$S_3 \sim \int d^3x \sqrt{-g}((1 + \frac{1}{2}\pi^0)R + \cdots), \quad (5.6)$$

and the Weyl transformation $\hat{H}_{\mu\nu} = h_{\mu\nu}^0 + \pi^0 g_{\mu\nu}^o$ is required to bring the action to Einstein frame.

5.2 Field equations to quadratic order

From the asymptotic expansion we now identify the fields of (5.1). In the asymptotic expansion we have retained only terms to quadratic order, that is of order $1/r$ and $1/r^2$ relative to the background. These terms are sufficient to determine vevs for the scalar chiral primaries of dimension one and two; the R symmetry currents and the energy momentum tensor. Using the tables in [33], one finds that the corresponding supergravity fields are $(s^1, s^2, \sigma^2, A_\mu^{1\pm}, H_{\mu\nu})$ respectively. Terms in other supergravity fields at the same order do not capture field theory data: they are simply induced by the non-linearity of the supergravity equations. Therefore we need only consider the above fields.

The next step is to derive the six-dimensional equations satisfied by the fluctuations, at non-linear order. The generic field equation for each field ψ^Q expanded in the number of fields is (schematically)

$$\mathcal{L}_Q \psi^Q = \mathcal{L}_{QRS}\psi^R\psi^S + \mathcal{L}_{QRST}\psi^R\psi^S\psi^T + \cdots, \quad (5.7)$$

where $\mathcal{L}_{Q_1\cdots Q_n}$ is generically a non-linear differential operator. (Note that each field ψ^Q should be the appropriate diffeomorphism invariant combination.) The complete set of corrections to the field equations involves many terms even to quadratic order.

Fortunately what is required for extracting field theory data is the equations of motion expanded perturbatively near the conformal boundary, where the radial coordinate acts as the perturbation parameter. This means that we need only retain terms on the right hand side which affect the radial expansion at sufficiently low order to impact on the vevs. In practice for our discussion, the relevant quadratic corrections are those involving two s^1 fields or two gauge fields, since all other quadratic terms do not contribute at the required order. (Note that there are no corrections involving one s^1 field and one gauge field.) That all other terms can be neglected will be justified when one carries out the holographic renormalization procedure and considers the perturbative solution of the field equations.

The scalar field corrections to the field equations were computed in [26, 27].[7] These computations along with the corrections quadratic in the gauge field are discussed in detail in appendix C. Consider first the scalar field equations. There are no quadratic corrections to the (s^1, s^2) equations from either s^1 fields or gauge fields, and thus the relevant equations remain the linearized equations. The σ^2 field equation does however get corrected by terms quadratic in scalars:

$$\Box \sigma_I^2 = \frac{11}{3}(s_i^1 s_j^1 - (D_\mu s_i^1)(D^\mu s_j^1))a_{Iij}. \tag{5.8}$$

The coefficient a_{Iij} is the triple overlap of the corresponding spherical harmonics (see appendix A). As discussed in the appendix C, there are also corrections to this equation quadratic in the gauge fields which involve the field strengths $F_{\mu\nu}(A^{\pm\alpha})$ associated with the connections $A_\mu^{\pm\alpha}$ respectively. However, according to the linearized field equations (5.4) these field strengths vanish and thus these corrections do not play a role.

Next consider the corrections to the Einstein equation, which are also discussed in more detail in C. Note that these corrections were not computed in [26, 27]. The appropriate three dimensional metric to quadratic order is

$$H_{\mu\nu} = h_{\mu\nu}^0 - \sum_{\alpha,\pm} h_\mu^{1\pm\alpha} h_\nu^{1\pm\alpha} + \pi^0 g_{\mu\nu}^o. \tag{5.9}$$

As discussed previously the quadratic term is necessary in order for the metric to transform correctly under diffeomorphisms. Then the equation satisfied by the metric, up to quadratic order in the scalar fields s_i^1 and the gauge fields is

$$(\mathcal{L}_E + 2)H_{\mu\nu} = 16(D_\mu s_i^1 D_\nu s_i^1 - g_{\mu\nu}^o s_i^1 s_i^1), \tag{5.10}$$

where the linearized Einstein operator was defined in (5.5). This equation can be rewritten as

$$\mathcal{G}_{\mu\nu} - g_{\mu\nu}^o = 16\left(D_\mu s_i^1 D_\nu s_i^1 - \frac{1}{2}g_{\mu\nu}^o((Ds_i^1)^2 - (s_i^1)^2)\right), \tag{5.11}$$

where $\mathcal{G}_{\mu\nu}$ is the linearized Einstein tensor. The rhs of this equation is the stress energy tensor of s^1. Note that the gauge field contributions to the energy momentum tensor involve the field strengths, and thus are zero when one imposes the lowest order field equation (5.4).

Finally, let us consider the equations for the gauge field. As discussed in [26, 27] the corrections quadratic in the gauge field correct the linearized equation to the non-Abelian Chern-Simons equation. That is, the six-dimensional equation is

$$\epsilon^{\mu\nu\rho}(\partial_\nu A_\rho^{\pm\alpha} + \frac{1}{2}A_\nu^{\pm\beta} A_\rho^{\pm\gamma}\epsilon_{\alpha\beta\gamma}) = 0, \tag{5.12}$$

where the $\epsilon_{\alpha\beta\gamma}$ arises from the triple overlap of vector harmonics defined in (A.17). Note that the $SU(2)_L$ and $SU(2)_R$ gauge fields are decoupled from each other. There are also corrections quadratic in the scalars s^1, which provide a source for the field strength:

$$\epsilon^{\mu\nu\rho}(\partial_\nu A_\rho^{\pm\alpha} + \cdots) = \pm 4s_i^1 D^\mu s_j^1 e_{\alpha ij}^\pm, \tag{5.13}$$

where the ellipses denote the non-linear Chern-Simons terms and the triple overlap is defined in (A.8).

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[7]We thank Gleb Arutyunov for making the latter available to us.

5.3 Reduction to three dimensions

Given the corrected six-dimensional field equations (5.8), (5.10) and (5.12), we now need to determine the corresponding three-dimensional field equations. As discussed in [20], the KK map between six and three dimensional fields is in general non-linear. The non-linear corrections arise from field redefinitions used to remove derivative couplings. From the form of the corrected field equations, it is apparent that only the scalar fields σ^2 are affected (at this order) by such field redefinitions. That is, the derivative couplings in (5.8) can be removed by the field redefinition

$$\Sigma_I^2 = \sqrt{32}\left(\sigma_I^2 + \frac{11}{6}s_i^1 s_j^1 a_{Iij} + \cdots\right), \tag{5.14}$$

where Σ_I^2 is the three dimensional field. (The prefactor ensures canonical normalization of the three dimensional field, as we will shortly discuss.) This field redefinition defines the KK reduction map between six and three dimensional fields.

The resulting set of three dimensional field equations can then be integrated to the following three-dimensional bulk action

$$\frac{n_1 n_5}{4\pi}\int d^3x \sqrt{-G}\left(R_G + 2 - \frac{1}{2}(DS_i^1)^2 + \frac{1}{2}(S_i^1)^2 - \frac{1}{2}(DS_I^2)^2 - \frac{1}{2}(D\Sigma_I^2)^2\right) \tag{5.15}$$

$$+\frac{n_1 n_5}{8\pi}\int\left(A_\alpha^+ dA^{+\alpha} + \frac{1}{3}\epsilon_{\alpha\beta\gamma}A^{+\alpha}A^{+\beta}A^{+\gamma} - A_\alpha^- dA^{-\alpha} - \frac{1}{3}\epsilon_{\alpha\beta\gamma}A^{-\alpha}A^{-\beta}A^{-\gamma}\right) + \cdots.$$

The ellipses denote fields dual to operators of higher dimension not being considered here, along with higher order interactions. The boundary terms in this action will be discussed later in the context of holographic renormalization.

An overall rescaling of the scalar fields arises from demanding that the three-dimensional scalar fields are canonically normalized, up to the overall scaling of the action; it follows from the quadratic actions given in [26]. Thus the three dimensional fields S_I^k and Σ_I^k are related to the six-dimensional fields s_I^k and σ_I^k via

$$S_I^k = 4\sqrt{k(k+1)}(s_I^k + \cdots), \quad \Sigma_I^k = 4\sqrt{k(k-1)}(\sigma_I^k + \cdots). \tag{5.16}$$

The ellipses denote non-linear terms in the KK map of which only (5.14) will be relevant here; other terms do not contribute to the order we need. The normalization of the gauge field terms also follows from the actions given in [26]. Note that the leading scalar field corrections to the gauge field equation (5.13) are also implicitly contained in the action (5.15), recalling that D is a covariant derivative and the scalar fields are charged under the SO(4) gauge group.

The overall prefactor in the action (5.15) follows from the chain of dimensional reductions

$$\frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-g_{10}}e^{-2\Phi}(R_{10}+\cdots) \rightarrow \frac{1}{2\kappa_6^2}\int d^6x\sqrt{-g}(R+\cdots) \rightarrow \frac{1}{2\kappa_3^2}\int d^3x\sqrt{-G}(R_G+2\cdots). \tag{5.17}$$

Implicitly in the latter expression the curvature scale is contained in the prefactor, so that the background AdS_3 metric G has unit radius. Then

$$2\kappa_{10}^2 = (2\pi)^7 (\alpha')^4; \quad 2\kappa_6^2 = \frac{1}{(2\pi)^4 V} 2\kappa_{10}^2; \quad 2\kappa_3^2 = \frac{1}{2\pi^2 Q_1 Q_5} 2\kappa_6^2, \tag{5.18}$$

which using (3.6) implies that

$$\frac{1}{2\kappa_3^2} = \frac{n_1 n_5}{4\pi}, \tag{5.19}$$

as in (5.15).

5.4 Holographic renormalization and extremal couplings

Having determined the three-dimensional fields and the equations of motion which they satisfy we are now ready to determine vevs using the procedure of holographic renormalization. We will first briefly review this procedure, using the Hamiltonian formalism developed in [22, 23]. Let $\mathcal{O}_{\Psi^k}$ be the dimension k operator dual to the three dimensional supergravity field Ψ^k, the latter being related to the six dimensional fields ψ^Q by non-linear KK maps. Then its vev can be expressed as

$$\langle \mathcal{O}_{\Psi^k} \rangle = \frac{n_1 n_5}{4\pi} \left((\pi_{\Psi^k})_{(k)} + \cdots \right); \tag{5.20}$$

where we will explain the meaning of the ellipses below. Now π_{Ψ^k} is the radial canonical momentum for the field Ψ^k and $(\pi_{\Psi^k})_{(k)}$ is the kth component in its expansion in terms of eigenfunctions of the dilatation operator. The results of [22, 23] show that there is a one to one correspondence between momentum coefficients and terms in the asymptotic expansion of the fields.

That is, the near boundary expansion of the metric and scalar fields is

$$ds_3^2 = \frac{dz^2}{z^2} + \frac{1}{z^2} \left(g_{(0)uv} + z^2 \left(g_{(2)uv} + \log(z^2) h_{(2)uv} + (\log(z^2))^2 \tilde{h}_{(2)uv} \right) + \cdots \right) dx^u dx^v;$$
$$\Psi^1 = z(\log(z^2) \Psi_{(0)}^1(x) + \tilde{\Psi}_{(0)}^1(x) + \cdots); \tag{5.21}$$
$$\Psi^k = z^{2-k} \Psi_{(0)}^k(x) + \cdots + z^k \Psi_{(2k-2)}^k(x) + \cdots, \quad k \neq 1.$$

In these expressions $(G_{(0)uv}, \Psi_{(0)}^1(x), \Psi_{(0)}^k(x))$ are sources for the stress energy tensor and scalar operators of dimension one and k respectively; as usual one must treat separately the operators of dimension $\Delta = d/2$, where d is the dimension of the boundary. Note that the 2-dimensional boundary coordinates are labeled by (u, v).

The correspondence between the momentum coefficients and these expansion coefficients for the scalar fields is then

$$(\pi_{\Psi^k})_{(k)} = ((2k - 2) \Psi_{(2k-2)}^k(x) + \cdots); \tag{5.22}$$
$$(\pi_{\Psi^1})_{(1)} = (2\tilde{\Psi}_{(0)}^1 + \cdots).$$

The ellipses denote non-linear terms in the relations that involve the sources and do not play a role here.

The ellipses in (5.20) denote terms non-linear in momenta. Such terms are related to extremal correlators and play a crucial role which we will discuss in detail. Before doing so, however, it is convenient to first discuss the gauge fields.

5.4.1 R symmetry currents

Let us now consider the vevs for R symmetry currents; these were previously discussed in [37, 38] and we will briefly summarize their results here. Given the asymptotic form of the metric (5.21) the Chern-Simons gauge fields have corresponding asymptotic field expansions

$$A^{\pm\alpha} = \mathcal{A}^{\pm\alpha} + z^2 A^{\pm\alpha}_{(2)} + \cdots. \tag{5.23}$$

Here $\mathcal{A}^{\pm\alpha}$ are fixed boundary values which are respectively holomorphic and anti-holomorphic. A key point is that the vev will be obtained from the leading order term in this expansion which is not affected by the other supergravity fields. Supergravity couplings affect only the subleading behavior of the gauge field, and thus we can neglect them. Put differently, the vev for the R symmetry current involves only the gauge field and there are no non-linear contributions.

The following boundary action

$$S_B = \frac{n_1 n_5}{16\pi} \int d^2 x \sqrt{-\gamma} \gamma^{uv} (\mathcal{A}^{+\alpha}_u \mathcal{A}^{+\alpha}_v + \mathcal{A}^{-\alpha}_u \mathcal{A}^{-\alpha}_v) \tag{5.24}$$

ensures that the variational problem for the gauge fields is well-defined with these boundary conditions; γ_{uv} is the induced boundary metric.[8] With these boundary terms the on-shell variation of the action yields the currents

$$\langle J^{\pm\alpha}_u \rangle = \frac{1}{\sqrt{-\gamma}} \left(\frac{\delta S}{\delta \mathcal{A}^{\pm\alpha u}_\alpha} \right) = \frac{n_1 n_5}{8\pi} (g_{(0)uv} \mp \epsilon_{uv}) \mathcal{A}^{\pm\alpha v}. \tag{5.25}$$

As discussed recently in [38] the resulting currents have the desired properties. In particular, momentarily switching to the Euclidean signature and using conformal gauge for the boundary metric so that $g_{(0)uv} dx^u dx^v = dw d\bar{w}$, the currents are

$$J^{+\alpha}_w = \frac{n_1 n_5}{4\pi} \mathcal{A}^{+\alpha}_w; \quad J^{+\alpha}_{\bar{w}} = 0; \tag{5.26}$$

$$J^{-\alpha}_w = 0; \quad J^{-\alpha}_{\bar{w}} = \frac{n_1 n_5}{4\pi} \mathcal{A}^{-\alpha}_{\bar{w}}.$$

Thus the $SU(2)_L$ and $SU(2)_R$ right currents are holomorphic and anti-holomorphic respectively, as expected for the boundary CFT. Moreover the current modes defined by

$$J^{+\alpha}_n = \frac{1}{2\pi i} \oint dw w^n J^{+\alpha}_w; \quad J^{-\alpha}_n = \frac{1}{2\pi i} \oint d\bar{w} \bar{w}^n J^{-\alpha}_{\bar{w}}, \tag{5.27}$$

obey the correct $SU(2)$ current algebras.

5.4.2 Scalar operators

Consider next the scalar operators; here the non-linear terms in (5.20) play a crucial role. Just as in [20] we need to take into account the rather subtle issue of extremal

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[8]In [37] the additional boundary term $\Delta S_{\mathcal{A}} = -\frac{n_1 n_5}{16\pi} \int d^2 x \sqrt{-\gamma} (\gamma^{uv} + \epsilon^{uv}) \mathcal{A}^{+\alpha}_u \mathcal{A}^{-\alpha}_v$ was added to the action. The variational problem is still consistent, but this term couples left and right movers so it is not appropriate for our purposes.

couplings. Recall that an extremal correlation function is one for which the dimension of one operator is equal to the sum of the other operator dimensions. The corresponding bulk couplings in supergravity vanish: this is physically necessary, because such couplings would induce conformal anomalies which are known to be zero (and non-renormalized). In [25] it was appreciated that extremal correlators are obtained not from bulk couplings, but instead from certain finite boundary terms. These would arise from demanding a well posed variational problem in the higher dimensional theory, and then keeping track of all boundary terms when carrying out the KK reduction.

These same extremal couplings play a key role in determining the vevs. Suppose the operator $\mathcal{O}_{\Psi^k}$ has a non-vanishing extremal n-point function with operators $\{\mathcal{O}_{\Psi^{k_a}}\}$, with $a = 1, \cdots (n-1)$. Then this implies an additional term in the holographic renormalization relation

$$\langle \mathcal{O}_{\Psi^k} \rangle = \frac{n_1 n_5}{4\pi} \bigg((\pi_{\Psi^k})_{(k)} + A_{kk_1 \cdots k_{(n-1)}} \prod_{k_a} (\pi_{\Psi^{k_a}})_{(k_a)} + \cdots \bigg) \qquad (5.28)$$

The coupling $A_{kk_1 \cdots k_{(n-1)}}$ must be such that one obtains the correct n-point function upon functional differentiation.

Now consider how this issue affects the vevs being determined here: there are potentially contributions to vevs of dimension two operators from their couplings to two dimension one operators. The latter include both the operators dual to the scalars S_i^1 and the R-symmetry currents dual to the gauge fields $A_\mu^{\pm\alpha}$. Let us consider first the following extremal three point functions between scalar operators

$$\Sigma^2 : \left\langle \mathcal{O}_{\Sigma_I^2} \mathcal{O}_{S_i^1} \mathcal{O}_{S_j^1} \right\rangle ; \quad S^2 : \left\langle \mathcal{O}_{S_I^2} \mathcal{O}_{S_i^1} \mathcal{O}_{S_j^1} \right\rangle . \qquad (5.29)$$

If these three point functions are non-zero, there will necessarily be additional quadratic contributions to the vevs of the dimension two operators.

In the discussions of [20] one could use the known free field extremal correlators of $\mathcal{N} = 4$ SYM along with non-renormalization theorems to fix the additional terms in (5.28). As we will discuss momentarily comparing with field theory is in this case rather more subtle. From the supergravity side there are two methods to compute these quadratic terms. The first would be to start with the six-dimensional action, demand that the variational problem is well-defined (which fixes boundary terms), and then dimensionally reduce to three dimensions. This is straightforward in principle, but to extract the required coefficient we need boundary terms cubic in the fields, which in turn requires expanding the field equations to cubic order. Thus we choose to use a second method: we compute the extremal correlator in supergravity by computing the corresponding non extremal correlator and then using a careful limiting procedure. This computation of the extremal correlators and hence the non-linear terms (5.28) is presented in appendix D.

Since all non-extremal three point functions between three $\mathcal{O}_{SI}$ operators vanish [36, 26], one also obtains no extremal three point function and therefore no extra contributions to $\langle S_I^2 \rangle$ beyond the standard term given in (5.20). The cubic coupling between one Σ field and two S fields is however generically non-vanishing [36, 26] and therefore we do obtain an extremal three point function which leads to the following result for the scalar

contributions to the one point function (D.12), (D.14)

$$\langle \mathcal{O}_{\Sigma_I^2} \rangle = \left(\frac{n_1 n_5}{4\pi} \right) \left(\pi_{(2)}^{\Sigma_I^2} - \frac{1}{4\sqrt{2}} a_{Iij} \pi_{(1)}^{S_i^1} \pi_{(1)}^{S_j^1} \right). \tag{5.30}$$

An extremal coupling between the dimension two scalar operators and two R symmetry currents would require a term in the rhs of (5.30) proportional to $\mathcal{A}_u \mathcal{A}^u$. However such term is gauge dependent and thus forbidden. We conclude that there are no additional contributions to (5.30).

Before leaving this section we should note why the extremal correlators were fixed via a limit of the non-extremal supergravity correlators and other indirect arguments rather than from a dual field theory computation. The relevant three point functions of scalar operators in the orbifold CFT were computed in [39] and [40]. There is no known non-renormalization theorem to protect them and thus no justification for extrapolating them to the strong coupling regime. Indeed, as we discuss in appendix F, certain correlation functions seem to disagree between supergravity and the orbifold CFT.

5.4.3 Stress energy tensor

Finally we discuss the vev for the stress energy tensor. This being a dimension two operator, we again need to take into account terms quadratic in two dimension one operators. Terms quadratic in the scalar fields S_i^1 and in the gauge fields $A_\mu^{\pm\alpha}$ both contribute. Let us momentarily suppress the gauge field contributions. Then as discussed in the previous section, the three dimensional metric couples at leading order to the scalar field S_i^1 in the three dimensional equations of motion and thus we need to derive the one point functions for this coupled system. This computation is very similar to the Coulomb branch analysis given in [41, 42] and is summarized in appendix E.

Next consider the additional contributions to the stress energy tensor quadratic in the gauge field. These immediately follow from the variation of the boundary terms (5.24), since the bulk Chern-Simons terms cannot contribute. Thus the total result for the stress energy tensor follows from (E.3) plus gauge field terms giving:

$$\langle T_{uv} \rangle = \frac{n_1 n_5}{2\pi} \left(g_{(2)uv} + \frac{1}{2} R g_{(0)uv} + \frac{1}{4} (\tilde{S}_{(0)}^1)^2 g_{(0)uv} + \frac{1}{4} (\mathcal{A}_{(u}^{+\alpha} \mathcal{A}_{v)}^{+\alpha} + \mathcal{A}_{(u}^{-\alpha} \mathcal{A}_{v)}^{-\alpha}) + \cdots \right), \tag{5.31}$$

where the terms in ellipses (source terms for the scalars) are given in (E.3) but do not contribute in our solutions. (Recall that parentheses denote the symmetrised traceless combination of indices.)

Now consider the effect of a large gauge transformation of the form $\mathcal{A}_w^{+3} \to \mathcal{A}_w^{+3} + \eta$. As discussed in [38] (see also [5]) this induces the shifts

$$L_0 \to L_0 + \eta J_0^{+3} + \frac{1}{4} k \eta^2; \quad J_0^{+3} \to J_0^{+3} + \frac{1}{2} k \eta, \tag{5.32}$$

where the Virasoro generator is defined as $L_0 = \frac{c}{24} + \oint dw T_{ww}$ and the level of the SU(2) algebra is $k \equiv n_1 n_5$. This is clearly a spectral flow transformation, and shows the relationship between bulk coordinate transformations on the S^3 and spectral flow in the boundary theory.

6. Vevs for the fuzzball solutions

We are now ready to extract the vevs from the asymptotic expansions of the fields in the fuzzball solutions given in (4.20), (4.21), (4.22) and (4.23). The appropriate (gauge invariant) combinations of six-dimensional scalar and gauge fields are

$$\mathbf{s_i^1} = \frac{1}{4r}(\mathbf{f_{1i}^1} - \mathbf{f_{1i}^5}) + \cdots; \quad \mathbf{s_I^2} = \frac{1}{8r^2}(\mathbf{f_{2I}^1} - \mathbf{f_{2I}^5}) + \cdots; \tag{6.1}$$

$$\sigma_I^2 = -\frac{1}{8r^2}(\mathbf{f_{2I}^1} + \mathbf{f_{2I}^5}) + \frac{1}{24r^2}(f_{1i}^5)(f_{1j}^5)a_{Iij} + \frac{1}{r^2}a^{\alpha-}a^{\beta+}f_{I\alpha\beta} + \cdots.$$

$$\mathbf{A_t^{+\alpha}} = -2\mathbf{a^{\alpha+}} + \cdots; \quad \mathbf{A_y^{+\alpha}} = 2\mathbf{a^{\alpha+}} + \cdots,$$

$$\mathbf{A_t^{-\alpha}} = -2\mathbf{a^{\alpha-}} + \cdots; \quad \mathbf{A_y^{-\alpha}} = -2\mathbf{a^{\alpha-}} + \cdots.$$

The graviton is given by

$$H_{tt} = f_{1i}^5 f_{1i}^5 - a^{\alpha+}a^{\alpha+} - a^{\alpha-}a^{\alpha-} + \cdots; \tag{6.2}$$

$$H_{yy} = -f_{1i}^5 f_{1i}^5 - a^{\alpha+}a^{\alpha+} - a^{\alpha-}a^{\alpha-} + \cdots;$$

$$H_{ty} = a^{\alpha+}a^{\alpha+} - a^{\alpha-}a^{\alpha-} + \cdots;$$

$$H_{rr} = -\frac{2}{r^4}f_{1i}^5 f_{1i}^5 + \cdots.$$

Next we extract the three-dimensional fields, which involves rescaling and shifting the scalar fields as defined in (5.14) and (5.16):

$$S_i^1 = -\frac{2\sqrt{2}}{r}f_{1i}^5 + \cdots; \quad S_I^2 = \frac{\sqrt{3}}{\sqrt{2}r^2}(f_{2I}^1 - f_{2I}^5) + \cdots; \tag{6.3}$$

$$\Sigma_2^I = \sqrt{32}\left(-\frac{1}{8r^2}(f_{2I}^1 + f_{2I}^5) + \frac{1}{2r^2}(f_{1i}^5)(f_{1j}^5)a_{Iij} + \frac{1}{r^2}a^{\alpha-}a^{\beta+}f_{I\alpha\beta} + \cdots\right).$$

where we used (4.10) in S_i^1. Note that the gauge fields and the metric are not rescaled or shifted upon the dimensional reduction to this order.

Thus for the scalar operators we obtain using (5.20) and (5.30) the vevs

$$\left\langle\mathcal{O}_{S_i^1}\right\rangle = \frac{n_1 n_5}{4\pi}(-4\sqrt{2}f_{1i}^5); \tag{6.4}$$

$$\left\langle\mathcal{O}_{S_I^2}\right\rangle = \frac{n_1 n_5}{4\pi}(\sqrt{6}(f_{2I}^1 - f_{2I}^5));$$

$$\left\langle\mathcal{O}_{\Sigma_I^2}\right\rangle = \frac{n_1 n_5}{4\pi}\sqrt{2}(-(f_{2I}^1 + f_{2I}^5) + 8a^{\alpha-}a^{\beta+}f_{I\alpha\beta}).$$

The currents follow from (5.25) as

$$\left\langle J^{+\alpha}\right\rangle = \frac{n_1 n_5}{2\pi}a^{\alpha+}(dy - dt); \quad \left\langle J^{-\alpha}\right\rangle = -\frac{n_1 n_5}{2\pi}a^{\alpha-}(dy + dt). \tag{6.5}$$

To evaluate the vev of the stress energy tensor using (5.31) we first need to bring the metric into the Fefferman-Graham coordinate system. This requires the following change of radial coordinate

$$z = \frac{1}{r} - \frac{1}{2r^3}(f_{1i}^5)^2 + \cdots \tag{6.6}$$

After changing radial coordinate in this way the metric becomes

$$ds_3^2 = \frac{dz^2}{z^2} + \frac{1}{z^2}(1 - 2(f_{1i}^5)^2 z^2)(-dt^2 + dy^2) \qquad (6.7)$$
$$-a^{\alpha+}a^{\alpha+}(dt - dy)^2 - a^{\alpha-}a^{\alpha-}(dt + dy)^2 + \cdots$$

The metric perturbation in the second line is traceless with respect to the leading order metric. Now applying the formula (5.31) we find that

$$\langle T_{uv} \rangle = 0. \qquad (6.8)$$

This is the anticipated answer, since these solutions are supposed to be dual to R vacua. The cancellation is however very non-trivial and needed all the machinery of holographic renormalization.

6.1 Higher dimension operators

Having extracted the vevs for all operators up to dimension two using the systematic procedure developed in [20], it is worth considering whether any predictions can be made for vevs of higher dimension operators. These could of course be determined by the same systematic procedure used above, by retaining all terms to sufficiently high order, but this would involve considerable computation.

It is therefore useful to recall at this point the result obtained in [35] for the vevs extracted from supergravity solutions corresponding to the Coulomb branch of $\mathcal{N} = 4$ SYM. When these solutions are asymptotically expanded in the radial coordinate of the defining harmonic function, non-linear terms in the vevs of CPOs arising from non-linear terms in the higher dimensional fields, non-linear terms in the KK reduction map and non-linear terms in the holographic renormalization relations all cancel out![9] The vevs are given by the *linear* terms in the higher dimensional fields. "Non-linear" in this context means terms which are non-linear in spherical harmonics.

Now consider what happens here if one retains only the linear terms in the fields, the dimensional reductions and the holographic renormalization relations. Then from (6.1), only the terms in boldface are retained. This means that there is no graviton perturbation to this order, and thus that the three-dimensional mass vanishes, in accordance with the expectation that these geometries describe R vacua. Furthermore, these terms give precisely the same results as before for the scalar $\mathcal{O}_S$ and current vevs, in which all non-linear contributions canceled. It is an interesting question to understand why the linear terms alone determine the stress energy tensor and $\mathcal{O}_S$ vevs. Note that just as in [35] a priori there is absolutely no justification for neglecting the non-linear terms, given that there is no small parameter. Presumably this question can be answered by understanding holographic renormalization directly in the higher dimension and developing the map between higher-dimensional fields and operators.

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[9]Strictly speaking, the cancellation was proven in [35] for operators of dimension four and less for which the corresponding vevs had been extracted using the rigorous procedures of [20]. However, the linearized approach gave results which agreed with the (non-renormalized) weak coupling field theory results for all dimension operators.

However, the linear terms clearly *fail* to give the correct answer for the operators dual to Σ^2. Thus the linearized approximation in this situation fails already at dimension two, which is the first place where non-linear terms can play a role (but note that it still holds for the dimension two operator $\mathcal{O}_{S^2}$).

Nevertheless one may proceed with the linearized procedure in order to get a rough idea of the behavior of the vevs for higher dimension operators. From the asymptotic expansion of the solution we extract the following linear terms for the scalars

$$s_I^k = \frac{1}{4kr^k}(f_{kI}^1 - f_{kI}^5) + \cdots \tag{6.9}$$
$$\sigma_I^k = -\frac{1}{4kr^k}(f_{kI}^1 + f_{kI}^5) + \cdots$$

From these asymptotics the vevs of the dual operators contain the linear terms

$$\left\langle \mathcal{O}_{S_I^k} \right\rangle = \left(\frac{n_1 n_5}{4\pi}\right) 2(k-1)\frac{\sqrt{k+1}}{\sqrt{k}}(f_{kI}^1 - f_{kI}^5 + \cdots); \tag{6.10}$$
$$\left\langle \mathcal{O}_{\Sigma_I^k} \right\rangle = -\left(\frac{n_1 n_5}{4\pi}\right) 2(k-1)\frac{\sqrt{k-1}}{\sqrt{k}}(f_{kI}^1 + f_{kI}^5 + \cdots),$$

where the ellipses denote the non-linear terms. Recall that (f_{kI}^1, f_{kI}^5) are proportional to the kth multipole moments of the D1 and D5 brane charge distributions, respectively. We will argue in the section 9 that these linear terms do not give the expected answer for the vevs of operators $\mathcal{O}_{\Sigma_I^k}$, although they seem to be sufficient to give the expected answer for the vevs of operators $\mathcal{O}_{S_I^k}$, at least for circular curves.

Following analogous arguments for the dimension k_v vector chiral primaries $J_{k_v}^{I_v \pm}$ dual to bulk vectors $A_{k_v}^{I_v \pm}$, we get the following structure

$$\left\langle J_{k_v}^{I_v \pm} \right\rangle \propto \left(\frac{n_1 n_5}{4\pi}\right) (A_{kI})_i E_{I_v I i}^{\pm}(dt \mp dy) + \cdots, \tag{6.11}$$

where the ellipses denote again the non-linear terms, the spherical harmonic triple overlap $E_{I_v I i}^{\pm}$ is defined in (A.10) and $(A_{kI})_i$ is defined in terms of the curve $F^i(v)$ in (4.6). To extract the exact coefficient relating the asymptotics of the bulk vector fields to the current vev, one would need to analyze the relevant Proca-Chern-Simons bulk equation and obtain the holographic renormalization relation for this case.

In the discussions of [35], the vevs obtained by the linearized approach gave correctly all the (non-renormalized) field theory vevs. Here the linearized approach does not give correctly vevs for chiral primaries. Moreover, we will also argue that there are additional vevs which are not captured by the linearized approximation at all. For example, when one linearizes the solution following the above procedure the (non primary) scalar fields (t_I^k, τ_I^k) are identically zero, but arguments given in section 9 suggest that the corresponding operators should in general have non-zero expectation values. Perhaps these vevs could still be extracted by an appropriate linearized analysis, but it is not apparent what the prescription should be. By contrast, the systematic method of [20] used in earlier sections will certainly give the correct answer for these vevs.

Note also that the linearized approximation manifestly gives different answers in different coordinate systems. For the example of the solution based on a circular curve we discuss in the next section, the linearized approximation in the coordinate system (7.10) actually gives the conjectured answers for scalar vevs, but linearizing in the hatted coordinate system (flat coordinates on R^4) gives different answers. Both in the fuzzball solutions considered here and in the Coulomb branch solutions discussed in [35] there are preferred coordinate systems, those in which the harmonic functions are naturally expressed. For the Coulomb branch this coordinate systems was precisely that in which linearizing gives the correct vevs, but here it does not. In general, however, there will be no preferred coordinate system or it may not be visible (as in (7.10)), and therefore there would be no natural way to linearize; one would have to apply the general methods of [20].

7. Examples

We discuss in this section the application of the general results to two specific examples: solutions based on circular and ellipsoidal curves, respectively.

7.1 Circular curves

A commonly used example of the fuzzball solutions is that in which the curve $F^i(v)$ is a (multiwound) circle [5, 6, 4],

$$F^1 = \mu_n \cos \frac{2\pi n v}{L}, \qquad F^2 = \mu_n \sin \frac{2\pi n v}{L}, \qquad F^3 = F^4 = 0. \tag{7.1}$$

The ten-dimensional solution in this case is conveniently written as

$$
\begin{aligned}
ds^2 = {}& f_1^{-1/2} f_5^{-1/2} \left(-\left(d\tilde{t} - \frac{\mu_n \sqrt{Q_1 Q_5}}{\hat{r}^2 + \mu_n^2 \cos^2 \hat{\theta}} \sin^2 \hat{\theta} d\phi \right)^2 + \left(d\tilde{y} - \frac{\mu_n \sqrt{Q_1 Q_5}}{\hat{r}^2 + \mu_n^2 \cos^2 \hat{\theta}} \cos^2 \hat{\theta} d\psi \right)^2 \right) \\
&+ f_1^{1/2} f_5^{1/2} \left((\hat{r}^2 + \mu_n^2 \cos^2 \hat{\theta}) \left(\frac{d\hat{r}^2}{\hat{r}^2 + \mu_n^2} + d\hat{\theta}^2 \right) + \hat{r}^2 \cos^2 \hat{\theta} d\psi^2 + (\hat{r}^2 + \mu_n^2) \sin^2 \hat{\theta} d\phi^2 \right) \\
&+ f_1^{1/2} f_5^{-1/2} dz \cdot dz;
\end{aligned} \tag{7.2}
$$

$$e^{2\Phi} = f_1 f_5^{-1},$$

$$f_{1,5} = 1 + \frac{Q_{1,5}}{\hat{r}^2 + \mu_n^2 \cos^2 \hat{\theta}},$$

whilst the tensor field is as in (3.1) with

$$A = \mu_n \frac{\sqrt{Q_1 Q_5}}{(\hat{r}^2 + \mu_n^2 \cos^2 \hat{\theta})} \sin^2 \hat{\theta} d\phi; \quad B = -\mu_n \frac{\sqrt{Q_1 Q_5}}{(\hat{r}^2 + \mu_n^2 \cos^2 \hat{\theta})} \cos^2 \hat{\theta} d\psi. \tag{7.3}$$

This solution is precisely of the form (3.1), using a non-standard coordinate system on R^4. That is, the hatted coordinates $(\hat{r}, \hat{\theta}, \phi, \psi)$ are related to usual coordinates (r, θ, ϕ, ψ) on R^4 such that the metric is

$$ds^2 = dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2 + \cos^2 \theta d\psi^2), \tag{7.4}$$

via

$$\hat{r} \cos \hat{\theta} = r \cos \theta; \quad r^2 = \hat{r}^2 + \mu_n^2 \sin^2 \hat{\theta}. \tag{7.5}$$

Note that this relation implies

$$\frac{1}{\hat{r}^2 + \mu_n^2 \cos^2 \hat{\theta}} = \frac{1}{\sqrt{(r^2 + \mu_n^2)^2 - 4\mu_n^2 r^2 \sin^2 \theta}}, \tag{7.6}$$

with the latter admitting the following asymptotic expansion

$$\frac{1}{\sqrt{(r^2 + \mu_n^2)^2 - 4\mu_n^2 r^2 \sin^2 \theta}} = \sum_{k \in 2Z} (-1)^{k/2} \frac{\mu_n^k Y_k^0(\theta_3)}{\sqrt{k+1} r^{2+k}}, \tag{7.7}$$

where the harmonic function is expanded in normalized spherical harmonics Y_k^0 which are singlets under the $SO(2)^2$ Cartan of $SO(4)$. These harmonics are given in (A.19); there is precisely one such singlet at each even degree. The asymptotic expansion in (7.7) follows from (4.4) upon using the fact that the lhs of (7.7) is equal to

$$\frac{1}{L} \int_0^L \frac{dv}{|x - F|^2}, \tag{7.8}$$

with F^i given in (7.1), so $\theta_3^F = \pi/2$ and $Y_k^0(\pi/2) = (-1)^{k/2} \sqrt{k+1}$.

The parameters (n, μ_n) labeling the curve are related to the charges via

$$n\mu_n = \frac{L}{2\pi} \sqrt{\frac{Q_1}{Q_5}} = \frac{\sqrt{Q_1 Q_5}}{R} \equiv \mu, \tag{7.9}$$

or equivalently $\mu_n = 1/(n\tilde{R})$, where $\tilde{R} = R/\sqrt{Q_1 Q_5}$. In deriving these results we have used (3.5) and (3.7).

The near horizon limit of (7.2) gives the six-dimensional fields

$$ds_6^2 = \sqrt{Q_1 Q_5} \left(-(\hat{r}^2 + \mu_n^2) dt^2 + \hat{r}^2 dy^2 + \frac{d\hat{r}^2}{(\hat{r}^2 + \mu_n^2)} \right) \tag{7.10}$$

$$+ \sqrt{Q_1 Q_5} \left(d\hat{\theta}^2 + \sin^2 \hat{\theta} (d\phi + \mu_n d\tilde{t})^2 + \cos^2 \hat{\theta} (d\psi - \mu_n d\tilde{y})^2 \right);$$

$$G^5 = \sqrt{Q_1 Q_5} \hat{r} dt \wedge dy \wedge d\hat{r} + \sqrt{Q_1 Q_5} \cos \hat{\theta} \sin \hat{\theta} d\hat{\theta} \wedge (d\phi + \mu_n dt) \wedge (d\psi - \mu_n dy).$$

with the scalar field Φ and the anti-self dual field G^6 vanishing. As previously, it is convenient to use the rescaled coordinates $\tilde{t} = \sqrt{Q_1 Q_5} t$ and $\tilde{y} = \sqrt{Q_1 Q_5} y$ so that the overall scale factor is manifest. Note that the coordinate y has periodicity $2\pi \tilde{R}$. When $n = 1$ there is a coordinate transformation $(\phi \to \phi + \mu_n t, \psi \to \psi + \mu_n y)$ that makes the metric exactly $AdS_3 \times S^3$. For $n > 1$ one can similarly shift the angular coordinates, but the resulting spacetime metric has a conical defect. As discussed in [5, 3], such a coordinate change is equivalent to carrying out a spectral flow to the NS sector; in the case of $n = 1$ the flow is to the vacuum. One way of seeing this is that under such a shift the Killing spinors change periodicity about the circle direction $\tilde{y}$. In the above coordinate system they are periodic, whilst after the coordinate transformation they are anti-periodic [5].

In the context of this paper, however, we are interested in R vacua of the CFT, and thus we do not wish to flow to the NS sector. This means we should interpret the solution in the original coordinate system, where the Killing spinors are periodic. From (7.10) we can immediately read off the three dimensional gauge field as

$$A^{-3} = \mu_n(dy + dt); \qquad A^{+3} = \mu_n(dy - dt). \tag{7.11}$$

The superscript indicates that the relevant Killing vectors are those given in the appendix in (A.14), such that A^{+3} and A^{-3} commute. The fact that there is a coordinate transformation where the solution is (locally) $AdS_3 \times S^3$ means that the three dimensional scalar fields $(S_i^1, S_I^2, \Sigma_I^2, \cdots)$ vanish. Note that the latter result is immediately obvious in the hatted coordinate system but it is not manifest in the coordinate system (r, θ, ϕ, ψ). That the S fields vanish in the latter coordinate system follows from (6.1) since $f_{kI}^1 = f_{kI}^5$. To see the vanishing of Σ_0^2 one has to use in (6.1) the identity

$$-\frac{1}{8}(f_{20}^1 + f_{20}^5) + f_{033}a^{3+}a^{3-} = 0, \tag{7.12}$$

which follows from (7.7) and the identity (A.18).

Now given the three dimensional fields we derive the corresponding vevs,

$$\langle T_{uv} \rangle = \left\langle \mathcal{O}_{S_i^1} \right\rangle = \left\langle \mathcal{O}_{S_I^2} \right\rangle = \left\langle \mathcal{O}_{\Sigma_0^2} \right\rangle = 0; \tag{7.13}$$

$$\langle J^{+3} \rangle = \frac{n_1 n_5}{4\pi}\mu_n(dy - dt); \qquad \langle J^{-3} \rangle = \frac{n_1 n_5}{4\pi}\mu_n(dt + dy).$$

Note that the R-symmetry charges

$$j^3 \equiv \int_0^{2\pi\tilde{R}} dy J_{\tilde{y}}^{+3} = \frac{n_1 n_5}{2n}; \tag{7.14}$$

$$\bar{j}^3 \equiv \int_0^{2\pi\tilde{R}} dy J_{\tilde{y}}^{-3} = \frac{n_1 n_5}{2n},$$

are quantized in half integral units provided that n is a divisor of $n_1 n_5$.

7.2 Ellipsoidal curves

The next simplest case to consider is a solution determined by a planar ellipsoidal curve:

$$F^1 = \mu_n a \cos \frac{2\pi n v}{L}, \qquad F^2 = \mu_n b \sin \frac{2\pi n v}{L}, \qquad F^3 = F^4 = 0, \tag{7.15}$$

with μ_n as in (7.9). The D1-brane charge constraint (3.5) implies that $(a^2 + b^2) = 2$. The vevs for this solution are given by

$$\langle T_{uv} \rangle = \left\langle \mathcal{O}_{S_i^1} \right\rangle = 0;$$

$$\langle J^{+3} \rangle = \frac{N}{4\pi}\mu_n ab(dy - dt); \qquad \langle J^{-3} \rangle = \frac{N}{4\pi}\mu_n ab(dt + dy);$$

$$\left\langle \mathcal{O}_{S_{m,\bar{m}}^2} \right\rangle = \left\langle \mathcal{O}_{\Sigma_{m,\bar{m}}^2} \right\rangle = 0; \qquad m \neq \bar{m} \tag{7.16}$$

$$\left\langle \mathcal{O}_{S^2_{1,1}} \right\rangle = \left\langle \mathcal{O}_{S^2_{-1,-1}} \right\rangle = -\frac{N}{8\sqrt{2}\pi}\mu_n^2(a^2-b^2);$$

$$\left\langle \mathcal{O}_{S^2_{0,0}} \right\rangle = \frac{N}{4\sqrt{2}\pi}\mu_n^2(a^2b^2-1);$$

$$\left\langle \mathcal{O}_{\Sigma^2_{1,1}} \right\rangle = \left\langle \mathcal{O}_{\Sigma^2_{-1,-1}} \right\rangle = -\frac{\sqrt{3}N}{8\sqrt{2}\pi}\mu_n^2(a^2-b^2);$$

$$\left\langle \mathcal{O}_{\Sigma^2_{0,0}} \right\rangle = \frac{\sqrt{3}N}{4\sqrt{2}\pi}\mu_n^2(a^2b^2-1).$$

Here we denote by $(m,\bar{m})$ the $(\mathrm{SU}(2)_L,\mathrm{SU}(2)_R)$ charges. The vanishing of the vevs of operators with charges $m \neq \bar{m}$ follows from the fact that the curve preserves rotational symmetry in the 3-4 plane. The equality of the vevs for operators with charge $(1,1)$ and $(-1,-1)$ follows from the orientation of the ellipse in the 1-2 plane: its axes are orientated with the 1-2 axes. Explicit representations of the corresponding spherical harmonics are given in (A.23).

One can also consider a planar ellipsoidal curve of different orientation, described by the curve

$$F^1 = \mu_n\left(a\cos\frac{2\pi nv}{L} + c\sin\frac{2\pi nv}{L}\right), \qquad F^2 = \mu_n\left(b\sin\frac{2\pi nv}{L} + d\cos\frac{2\pi nv}{L}\right), \quad (7.17)$$

with $F^3 = F^4 = 0$ and μ_n as in (7.9). The D1-brane charge constraint (3.5) in this case requires that $(a^2 + b^2 + c^2 + d^2) = 2$. The non-vanishing vevs are

$$\left\langle J^{+3} \right\rangle = \frac{N}{4\pi}\mu_n(ab-cd)(dy-dt); \qquad \left\langle J^{-3} \right\rangle = \frac{N}{4\pi}\mu_n(ab-cd)(dt+dy);$$

$$\left\langle \mathcal{O}_{S^2_{\pm1,\pm1}} \right\rangle = -\frac{N}{8\sqrt{2}\pi}\mu_n^2((a\pm id)^2 + (c\pm ib)^2);$$

$$\left\langle \mathcal{O}_{S^2_{0,0}} \right\rangle = \frac{N}{4\sqrt{2}\pi}\mu_n^2((ab-cd)^2 - 1); \qquad\qquad (7.18)$$

$$\left\langle \mathcal{O}_{\Sigma^2_{\pm1,\pm1}} \right\rangle = -\frac{\sqrt{3}N}{8\sqrt{2}\pi}\mu_n^2((a\pm id)^2 + (c\pm ib)^2);$$

$$\left\langle \mathcal{O}_{\Sigma^2_{0,0}} \right\rangle = \frac{\sqrt{3}N}{4\sqrt{2}\pi}\mu_n^2((ab-cd)^2 - 1).$$

The vevs for operators with charge $(1,1)$ and $(-1,-1)$ are no longer equal, since the axes of the ellipse are no longer orientated with the 1-2 axes. The vevs are however complex conjugate, as they must be since the operators are complex conjugate to each other.

8. Dual field theory

To understand the interpretation of the holographic results it will be useful to review certain aspects of the dual CFT and the ground states of the R sector. The dual CFT is believed to be a deformation of the $\mathcal{N} = (4,4)$ supersymmetric sigma model with target space $S^N(X)$, where $N = n_1 n_5$ and the compact space is either T^4 or $K3$. Most of the discussion below will be for the case of T^4, although the results extend simply to $K3$. The

orbifold point is roughly the analogue of the free field limit of $\mathcal{N} = 4$ SYM in the context of AdS_5/CFT_4 duality.

The chiral primaries and R ground states can be precisely described at the orbifold point. In particular, there exists a family of chiral primaries in the NS-NS sector associated with the $(0,0)$, $(2,0)$, $(0,2)$, $(1,1)$ and $(2,2)$ cohomology of the internal manifold (we do not discuss the chiral primaries associated with odd cohomology in this paper). These can be labeled as

$$\mathcal{O}_n^{(0,0)}, \quad h = \bar{h} = \frac{1}{2}(n-1); \tag{8.1}$$

$$\mathcal{O}_n^{(2,0)}, \quad h = \bar{h} + 1 = \frac{1}{2}(n+1);$$

$$\mathcal{O}_n^{(1,1)q}, \quad h = \bar{h} = \frac{1}{2}n; \qquad q = 1, \ldots, h^{1,1}$$

$$\mathcal{O}_n^{(0,2)}, \quad h = \bar{h} - 1 = \frac{1}{2}(n-1);$$

$$\mathcal{O}_n^{(2,2)}, \quad h = \bar{h} = \frac{1}{2}(n+1),$$

where n is the twist, $h^{1,1}$ in the dimension of the $(1,1)$ cohomology and $h = j^3$, $\bar{h} = \bar{j}^3$. The operator $\mathcal{O}_1^{(0,0)}$ is the identity operator. The complete set of chiral primaries associated with this cohomology is built from products of the form

$$\prod_{l=1} (\mathcal{O}_{n_l}^{(p_l+1,q_l+1)})^{m_l}, \qquad \sum_{l=1}^{I} n_l m_l = N, \tag{8.2}$$

where p_l, q_l take the values ± 1 (so that one gets the product of operators in (8.1); we suppress the index q) and symmetrization over the N copies of the CFT is implicit.

In [43] the spectrum of chiral primary operators of the orbifold CFT was matched with the KK spectrum. One should note however that the matching is not canonical in the sense that the operators at the orbifold point and the fields in supergravity are characterized by additional labels not visible in the other description. In particular, the supergravity spectrum is also organized in representations of an additional[10] $\widetilde{SO}(4) \times SO(n_t)$, as can be seen from the tables of [33], where the $\widetilde{SO}(4)$ is the R-symmetry of the 6D supergravity (not to be confused with the SO(4) R-symmetry of the CFT which is related to the isometries of the S^3) and n_t is the number of tensor multiplets. On the other hand, the chiral spectrum at the orbifold point is characterized by the set of integers n_l, m_l and the type of operator associated with these, as in (8.2). Furthermore, there is an additional $SO(4)_I$ acting on the chiral spectrum, related to global rotations of T^4 (see, for example, [39] or the review [44]). It is not immediately clear how the labels n_l, m_l translate in the supergravity description and what is the relation of $SO(4)_I$ with the supergravity $\widetilde{SO}(4) \times SO(5)$ ($n_t = 5$ for T^4).

To get a more precise mapping let us consider the special case of chiral primaries with $h = \bar{h}$. We see from (8.1) that there are 6 such operators for any $h < N/2$, except when $h = 1/2$ in which case there are only 5 operators ($\mathcal{O}_1^{(0,0)}$ is the identity operator). In all

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[10]$\widetilde{SO}(4)$ was called SO(4)$_R$ in [33].

cases 4 of these operators form a vector of $SO(4)_I$. On the supergravity side, the fields $S_k^{(r)I}$ and Σ_k^I have the correct dimensions and charges to correspond to these operators. Note that $k > 1$ for Σ_k^I, so we indeed have only 5 fields corresponding to operators of dimension $(1/2, 1/2)$. These fields are singlets under $\widetilde{SO}(4)$ and $S_k^{(r)I}$ transforms in the vector of $SO(5)$. It thus appears natural to identify $SO(4)_I$ with an $SO(4)$ subgroup of $SO(5)$ and to make the correspondence

$$S_n^{p(q+6)} \quad \leftrightarrow \quad \mathcal{O}_n^{(1,1)q}, \quad q = 1, \ldots, 4, \quad n \geq 1 \tag{8.3}$$

where here and below the superscript p denotes that the relevant scalar fields are those for which $j^3 = j$ and $\bar{j}^3 = \bar{j}$. The question is then whether $\mathcal{O}_{n+1}^{(0,0)}$ or $\mathcal{O}_{n-1}^{(2,2)}$ corresponds to $S_n^{p(6)}$. The most natural correspondence seems to be

$$\begin{aligned} S_n^{p(6)} \quad &\leftrightarrow \quad \mathcal{O}_{n+1}^{(0,0)}, \quad n \geq 1; \\ \Sigma_n^p \quad &\leftrightarrow \quad \mathcal{O}_{n-1}^{(2,2)} \quad n \geq 2. \end{aligned} \tag{8.4}$$

This identification is natural given that there is no Σ_1 in supergravity but is clearly not unique because S_n^p and Σ_n^p have the same charges so it could be that different combinations of them correspond to the operators at the orbifold point.

A similar discussion holds for chiral primaries with $h - \bar{h} = \pm 1/2, \pm 1$. The case of $h - \bar{h} = \pm 1/2$ is not relevant here since we are not considering solutions associated with odd cohomology in this paper. The case $h - \bar{h} = \pm 1$ is relevant but most of the points we want to make can be made using examples that utilize only chiral primaries with $h = \bar{h}$, so we will not need a detailed discussion of them. We only mention that the corresponding supergravity fields are massive vector fields.

Spectral flow maps these chiral primaries in the NS sector to R ground states, where

$$\begin{aligned} h^R &= h^{NS} - j_3^{NS} + \frac{c}{24}; \\ j_3^R &= j_3^{NS} - \frac{c}{12}, \end{aligned} \tag{8.5}$$

where c is the central charge. Each of the operators in (8.2) is mapped by spectral flow to an operator of definite R-charge

$$\prod_{l=1}(\mathcal{O}_{n_l}^{(p_l+1,q_l+1)})^{m_l} \quad \rightarrow \quad \mathcal{O}^{R(2j_3^R, 2\bar{j}_3^R)}, \qquad j_3^R = \frac{1}{2} \sum_l p_l m_l, \; \bar{j}_3^R = \frac{1}{2} \sum_l q_l m_l. \tag{8.6}$$

In particular, for fixed twist n the operators in (8.1) have the following charges after the flow

$$\begin{aligned} \mathcal{O}_n^{(0,0)} &\rightarrow \mathcal{O}_n^{R(-,-)}; \\ \mathcal{O}_n^{(2,0)} &\rightarrow \mathcal{O}_n^{R(+,-)}; \\ \mathcal{O}_n^{(0,2)} &\rightarrow \mathcal{O}_n^{R(-,+)}; \\ \mathcal{O}_n^{(2,2)} &\rightarrow \mathcal{O}_n^{R(+,+)}; \\ \mathcal{O}_n^{(1,1)q} &\rightarrow \mathcal{O}_n^{R(0,0)q}, \end{aligned} \tag{8.7}$$

where it is understood that each of these operators is tensored by the appropriate power of the identity operator such that (8.2) holds. For example, $\mathcal{O}_n^{(0,0)}$ should be tensored by $(\mathcal{O}_1^{(0,0)})^{N-n}$, and the R-symmetry charge of the flown operator $\mathcal{O}_n^{R(-,-)}$ follows from (8.5) with $c = 6n$. It follows from (8.7) that the operators $\mathcal{O}_n^{R(\pm,\pm)}$ form a $(\frac{1}{2}, \frac{1}{2})$ representation of $SU(2)_L \times SU(2)_R$ whilst the operators $\mathcal{O}_n^{R(0,0)q}$ are q singlets. From the form of the operators in the NS sector (8.2) it is clear that $j^R \leq \frac{1}{2}N$, since one can have at most N operators in the product. Symmetrization over the copies of the CFT means that spectral flow in the left and right moving sectors is not quite independent. When one has m copies of the same operator one needs to symmetrize over copies and thus one obtains only states with $j^R = \bar{j}^R = \frac{1}{2}m$ (although the values of j_3^R and $\bar{j}_3^R$ range independently from $-j^R$ to j^R).

We will label by the R-charges the states obtained by the usual operator-state correspondence,

$$|j_3^R, \bar{j}_3^R\rangle = \mathcal{O}^{R(2j_3^R, 2\bar{j}_3^R)}(0)|0\rangle. \tag{8.8}$$

8.1 R ground states and vevs

The R ground states can also be characterized by the expectation value of gauge invariant operators in them. Since the fuzzball solutions are conjectured to be dual to R ground states and the vevs of gauge invariant operators is the information we extracted from the fuzzball solutions we would like to see what one can say about them using the dual CFT. There are two sets of constraints on these vevs: kinematical and dynamical.

8.1.1 Kinematical constraints

The kinematical constraints follow from symmetry considerations and they have been recently discussed in [15]. As discussed above the R ground states in the (usual) basis are eigenstates of the R-symmetry charge. This implies that only neutral operators can have a non-vanishing vev,

$$\langle -j_3^R, -\bar{j}_3^R | O^{(k_1, k_2)} | j_3^R, \bar{j}_3^R \rangle = 0, \qquad \{k_1 \neq 0 \ or \ k_2 \neq 0\} \tag{8.9}$$

where k_1 and k_2 are the R-charges of the operator and we use the fact that the bra state has the opposite R charge to the ket state.

8.1.2 Dynamical constraints and 3-point functions

The vevs of neutral gauge invariant operators are determined dynamically. One way to determine them is using 3-point functions at the conformal point. Let $|\Psi\rangle = O_\Psi(0)|0\rangle$. Then the vev of an operator O_k of dimension k in the this state is given by

$$\langle \Psi | O_k(\lambda^{-1}) | \Psi \rangle = \langle 0 | (O_\Psi(\infty))^\dagger O_k(\lambda^{-1}) O_\Psi(0) | 0 \rangle, \tag{8.10}$$

where λ is a mass scale. For scalar operators the 3-point function is uniquely determined by conformal invariance and the above computation yields

$$\langle \Psi | O_k(\lambda^{-1}) | \Psi \rangle = \lambda^k C_{\Psi k \Psi} \tag{8.11}$$

where $C_{\Psi k\Psi}$ is the fusion coefficient. Similarly, the expectation value of a symmetry current measures the charge of the state

$$\langle\Psi|j(\lambda^{-1})|\Psi\rangle = \langle 0|(O_\Psi(\infty))^\dagger j(\lambda^{-1}) O_\Psi(0)|0\rangle = q\lambda\langle\Psi|\Psi\rangle \tag{8.12}$$

where q is the charge of the operator O_Ψ under j.

Let us now apply these principles to the cases of interest here. We will thus need to know the 3-point functions at the conformal point, which can be computed in the NS sector and then flowed to the R sector. A computation of 3-point functions at the orbifold point has been given in [39, 40]. We however need to know the result in the regime where supergravity is valid. For the theory at hand there is no known non-renormalization theorem that would protect the 3-point functions. Moreover, as discussed in appendix F, the 3-point functions that can also be computed holographically (i.e. those involving only operators dual to supergravity fields) are different from the 3-point functions computed at the orbifold point.

So the only dynamical tests that one can currently do must involve states created by operators corresponding to single particle states. In our case the fuzzball solutions are meant to correspond to the R ground states connected with universal cohomology, so only states created by the operators $\mathcal{O}_n^{R(\pm,\pm)}$ are relevant. For these cases the corresponding 3-point point functions can be computed by standard holographic methods using the results in [36, 26].

Let $\Phi = (S, A^+, A^-, \Sigma)$ be the fields dual to the operators $\mathcal{O}_n^{R(\pm,\pm)}$. The three point functions involving scalar chiral primaries have the following structure

$$\left\langle \mathcal{O}_\Phi^\dagger \mathcal{O}_\Sigma \mathcal{O}_\Phi \right\rangle \neq 0, \quad \left\langle \mathcal{O}_\Phi^\dagger \mathcal{O}_S \mathcal{O}_\Phi \right\rangle = 0. \tag{8.13}$$

where $\mathcal{O}_\Phi^\dagger$ denotes the conjugate operator with $j^3 = -j$, $\bar{j}^3 = -\bar{j}$. Our results for the vevs include the lowest dimension operators in these towers.

From the results of [26] there are however other non-zero three point functions in supergravity, such as

$$\left\langle \mathcal{O}_\Phi^\dagger \mathcal{O}_\tau \mathcal{O}_\Phi \right\rangle \neq 0, \quad \left\langle \mathcal{O}_\Phi^\dagger \mathcal{O}_{\rho^\pm} \mathcal{O}_\Phi \right\rangle \neq 0, \quad \left\langle \mathcal{O}_\Phi^\dagger \mathcal{O}_{A^\pm} \mathcal{O}_\Phi \right\rangle \neq 0, \quad \cdots \tag{8.14}$$

where the ellipses denote other operators, dual to other vectors and KK gravitons. These operators all have sufficiently high dimensions that we did not compute their vevs. Moreover, the vevs of these operators are not captured at all by the linearized approximation.

9. Correspondence between fuzzballs and chiral primaries

9.1 Correspondence with circular curves

Having reviewed the description of the degenerate R ground states in the CFT we now turn to the connection with the fuzzball solutions. The basic proposal is that there is a correspondence between the R ground states and the curves $F^i(v)$ defining the supergravity solutions. Let us consider first states of the specific form

$$(\mathcal{O}_n^{R(\pm,\pm)})^{\frac{N}{n}}|0\rangle, \qquad j_3^R = \pm\frac{N}{2n}; \quad \bar{j}_3^R = \pm\frac{N}{2n}. \tag{9.1}$$

Then such ground states are proposed to be in one to one correspondence with circular curves [1]:

$$(\mathcal{O}_n^{R(+,+)})^{\frac{N}{n}}|0\rangle \quad \leftrightarrow \quad F^1 = \frac{\mu}{n}\cos\left(\frac{2\pi nv}{L}\right); \quad F^2 = \frac{\mu}{n}\sin\left(\frac{2\pi nv}{L}\right), \qquad (9.2)$$

with $F^3 = F^4 = 0$ and where the parameter μ is fixed via (3.5) to be $\sqrt{Q_1 Q_5}/R$, see (7.9). Similarly $(\mathcal{O}_n^{R(-,-)})^{N/n}$ corresponds to a circle of the same radius in the 1-2 plane with the opposite rotation (that is, $F^2 \to -F^2$) and the operators $(\mathcal{O}_n^{R(+,-)})^{N/n}, (\mathcal{O}_n^{R(-,+)})^{N/n}$ correspond to circles in the 3-4 plane.

Note the states (9.1) are generically not dual to supergravity fields. Only the specific states obtained by flowing the NS operators $((\mathcal{O}_1^{(0,0)})^N, \mathcal{O}_N^{(p,q)})$ correspond to supergravity fields. All product operator do not correspond to supergravity fields, with the exception of $(\mathcal{O}_1^{(0,0)})^N$, since this is simply the identity operator in the NS sector. Moreover, whilst the operators $\mathcal{O}_N^{(p,q)}$ are dual to supergravity fields their special properties (following from having maximal dimension) are not visible in supergravity computations which effectively takes $N \to \infty$.

There are various pieces of evidence for this correspondence between states and circular curves. Firstly the rotation charges match, using the discussions in section 7.1, in particular (7.14). Secondly, as first discussed in [1], one can consider absorption processes in the corresponding geometries, and compare the scattering behavior with CFT expectations; they agree. (Note that for a general fuzzball geometry the wave equation for minimal scalars is not separable, so the absorption cross-section cannot be computed, and this comparison cannot be made.)

Our results for the scalar 1-point functions in (7.13) (along with (6.10)) give more data which can be used to test the proposed correspondence. As discussed previously kinematical constraints arise simply from charge conservation: if the R ground state is an eigenstate of both j_3^R and $\bar{j}_3^R$ then only scalar operators with $j^3 = \bar{j}^3 = 0$ can acquire a vev. These correspond to the Y_0^k harmonics discussed in section 7.1. Thus the fact that only such operators appear in (7.13) follows solely from kinematics.

Determining which of the (kinematically allowed) operators actually acquire a vev involves dynamics also and is rather more subtle. Consider first the special case where the operator (8.2) determining the ground state is the product $(\mathcal{O}_1^{(0,0)})^N$, that is, the NS vacuum. Then clearly all three point functions vanish, and thus all 1-point functions (apart from j) in the corresponding R vacuum must vanish.

Moreover the vanishing of all 1-point functions implies that the non-linear terms in the vevs of $\mathcal{O}_{\Sigma_I^k}$ in (6.10) must contribute. The linear terms in (6.10) do give the expected vanishing vev for $\mathcal{O}_{S_I^k}$ since the D1-brane and D5-brane densities are constant along the curve. However, for the circular profile the linear terms in the $\mathcal{O}_{\Sigma_I^k}$ vevs following from (6.10) give

$$\left\langle \mathcal{O}_{\Sigma_0^k} \right\rangle = (-)^{k+1/2} N \left(\frac{\sqrt{Q_1 Q_5}}{R}\right)^k \frac{(k-1)^{3/2}}{\pi\sqrt{k(k+1)}} + \cdots \qquad (9.3)$$

and therefore the non-linear terms denoted by ellipses must contribute, to give the expected zero vev.

Next consider the cases where the operator (8.2) determining the ground state is $(\mathcal{O}_1^{(2,0)})^N$, $(\mathcal{O}_1^{(0,2)})^N$ or $(\mathcal{O}_1^{(2,2)})^N$. The supergravity solutions corresponding to these vacua are clearly closely related to that just discussed: the defining curve is still a circle with radius $a = 1/\tilde{R}$, but the rotation is in the opposite direction or the circle lies in the 3-4 plane. Therefore the one point functions should also vanish in these three cases. This is consistent with the fact that these NS operators are related to the NS vacuum under spectral flow by an integral parameter (i.e. NS to NS). That is, under a spectral flow

$$h' = h - 2\theta j + \frac{c\theta^2}{6}; \quad j_3' = j_3 - \frac{c\theta}{6} \tag{9.4}$$

with $\theta = 1$ the chiral primary with maximal $j^3 = N$ is mapped to the vacuum.

Now let us move to the more general states of the form (9.1), which are conjectured to correspond to circular curves. Still there are no scalar chiral primary vevs according to (9.3). Kinematics again dictates that only $j^3 = \bar{j}^3 = 0$ operators acquire a vev, but the fact that kinematically allowed vevs are zero follows from dynamical information about three point functions. In particular, one needs to know the three point functions at the conformal point for operators $\mathcal{O}_\Phi$ which are products in the CFT, and which therefore do not correspond to single particle supergravity fluctuations. These are not known, so the results for the vevs provide a prediction for these correlation functions at strong coupling, provided the conjectured correspondence is correct.

9.2 Non-circular curves

Next we consider the curves corresponding to the most general states of the form (8.2); it has been conjectured that these should correspond to connected curves in R^4. For example, a state of the form

$$(\mathcal{O}_n^{R++})^{\gamma N/n}(\mathcal{O}_n^{R--})^{\delta N/n} \quad \gamma + \delta = 1 \quad j_3^R = \bar{j}_R^3 = \frac{1}{2}N(\gamma - \delta)/n, \tag{9.5}$$

was conjectured in [1] to correspond to an elliptical curve

$$F^1(v) = \mu\frac{a}{n}\cos\left(\frac{2\pi nv}{L}\right); \quad F^2(v) = \mu\frac{b}{n}\sin\left(\frac{2\pi nv}{L}\right), \tag{9.6}$$

with $F^3 = F^4 = 0$ and $\mu = \sqrt{Q_1 Q_5}/R$. Provided that

$$a = \frac{1}{\sqrt{2}}(\sqrt{1 + (\gamma - \delta)} + \sqrt{1 - (\gamma - \delta)}); \quad b = \frac{1}{\sqrt{2}}(\sqrt{1 + (\gamma - \delta)} - \sqrt{1 - (\gamma - \delta)}), \tag{9.7}$$

the supergravity solution would have the correct angular momenta to match with the field theory state.

Without any further data to match between supergravity and field theory one could not check the proposed correspondence further. The one point functions of chiral primaries computed here, however, immediately contradict the correspondence between operators of the form (8.2) and connected curves in R^4. The issue is the following. States of the form (8.2) are eigenstates of angular momentum operators j_R^3 and $\bar{j}_R^3$. This means that

scalar operators can acquire a vev only if $j_R^3 = \bar{j}_R^3 = 0$, following (8.9). Note that this is again purely kinematical, with dynamical information determining precisely which of these operators actually acquire a vev.

However, the supergravity solution generated by a connected curve will, according to the formulae, give rise to non-zero vevs for operators with $(j_R^3, \bar{j}_R^3) \neq 0$ whenever the curve is not circular. Put differently, a non-circular curve explicitly breaks the $SO(2) \times SO(2)$ symmetries, with the symmetry breaking characterized by the vevs for operators with non-zero $(j_R^3, \bar{j}_R^3)$.

One might wonder whether a non-circular curve could nonetheless give rise to vevs only for $j_R^3 = \bar{j}_R^3 = 0$ operators. That is, although the curve is non-circular in flat coordinates on R^4, it might be circular in another coordinate system, and the vevs might be related to multipole moments in that coordinate system. This however contradicts the explicit formulae for the vevs, exemplified by the case of an ellipsoidal curve, whose vevs are given in (7.16). More generally, the vevs will clearly involve the multipole moments of the charge distribution on the R^4.

9.3 Testing the new proposal

Now consider the proposal made in [15] and here, that the supergravity solution defined by a given curve is dual to a linear superposition of states with coefficients following from those in the coherent state in the dual FP system. In particular, according to (1.13) and (2.29) the ellipse (9.6) would be dual to the linear superposition

$$|ellipse) = \sum_{k=0}^{N/n} \frac{1}{2^{\frac{N}{n}}} \sqrt{\frac{(\frac{N}{n})!}{(\frac{N}{n}-k)!k!}} (a+b)^{\frac{N}{n}-k} (a-b)^k (\mathcal{O}_n^{R++})^{(\frac{N}{n}-k)} (\mathcal{O}_n^{R--})^k; \qquad (9.8)$$

note that $(a^2 + b^2) = 2$ and that (a, b) are both real.

The issue is whether this proposal is consistent with the vevs extracted from the corresponding geometry in section (7.2). Again this question is divided into kinematical and dynamical parts. The fact that operators with equal and opposite J^{12} charge acquire equal values in section (7.2) follows from the orientation of the ellipse and is a kinematical constraint which must also be implicit in the dual description. (That operators with non-zero J^{34} charge do not acquire a vev is also a kinematical constraint, of course, but this is automatically satisfied for any proposed dual involving only operators of zero J^{34} charge.) The actual non-zero values for the vevs in section (7.2) require dynamical information.

So does the proposed linear superposition satisfy the kinematical constraints? We can prove that it does as follows. Let us write (9.8) as

$$|ellipse) = \sum_{k=0}^{N/n} a_k |\left(\frac{N}{n} - k\right); k\rangle, \qquad (9.9)$$

where $|(\frac{N}{n} - k); k\rangle$ is shorthand for the state created by $(\mathcal{O}_n^{R++})^{(\frac{N}{n}-k)} (\mathcal{O}_n^{R--})^k$ and a_k are real coefficients (that can be read-off from (9.8)). Now consider a general J^{12} charged

operator $\mathcal{O}_{m,m}$. Its vev is given by

$$(ellipse|\mathcal{O}_{m,m}|ellipse) = \sum_{k=0}^{N/n-m} a_k^* a_{m+k} \langle \left(\frac{N}{n}-k\right);k|\mathcal{O}_{m,m}|\left(\frac{N}{n}-k-m\right);k+m\rangle, \quad (9.10)$$

whilst the corresponding operator with opposite charge $\mathcal{O}_{-m,-m}$ acquires a vev

$$(ellipse|\mathcal{O}_{-m,-m}|ellipse) = \sum_{k=0}^{N/n-m} a_{m+k}^* a_k \left(\langle \left(\frac{N}{n}-k\right);k|\mathcal{O}_{m,m}|\left(\frac{N}{n}-k-m\right);k+m\rangle\right)^{\dagger}, \quad (9.11)$$

Given that the coefficients a_m are real, the vevs (9.10) and (9.11) will be the same provided that the overlaps are real; the fusion coefficients for the corresponding extremal three point functions do indeed have this property.

To test the values of the non-zero vevs in (7.16) one needs dynamical information. One can check that the R charges are in agreement with those of the superposition (9.8) as follows. The state $|(\frac{N}{n}-k);k\rangle$ is an eigenstate of both j^3 and $\bar{j}^3$ with (equal) eigenvalues $(N/2n-k)$. Then

$$(ellipse|j^3|ellipse) = \sum_{k=0}^{N/n} \frac{1}{2^{2\frac{N}{n}}} \frac{(\frac{N}{n})!}{(\frac{N}{n}-k)!k!}(a+b)^{2\frac{N}{n}-k}(a-b)^{2k}\left(\frac{N}{2n}-k\right) \quad (9.12)$$

$$= -\frac{(a^2-b^2)^{\frac{N}{n}}}{2^{\frac{2N}{n}+1}}z\frac{\partial}{\partial z}\left(z+\frac{1}{z}\right)^{\frac{N}{n}} = \frac{N}{2n}ab; \quad z = \frac{(a-b)}{(a+b)};$$

with the same result for $\bar{j}^3$. This is in exact agreement with the result of (7.16).

The remaining non-zero vevs of (7.16) are the vevs of the charged operators $\mathcal{O}_{1,1} \equiv \{\mathcal{O}_{S_{1,1}}, \mathcal{O}_{\Sigma_{1,1}}\}$, and the neutral operators, $\mathcal{O}_{0,0} \equiv \{\mathcal{O}_{S_{0,0}}, \mathcal{O}_{\Sigma_{0,0}}\}$, where (m,n) denote the $(\mathrm{SU}(2)_L, \mathrm{SU}(2)_R)$ charges. To test whether the proposal is consistent with these vevs is far more difficult: we would need to know the three point functions between all operators occurring in (9.8) and the dimension two operators. Given that the former are not dual to supergravity fields, we do not have any information about the relevant three point functions and thus cannot check the vevs. That said, a well motivated guess for the structure of the three point functions leads to vevs which agree remarkably well with those in (7.16).

Note that in (7.16) the vevs of the operators with the same charges are the same up to overall numerical coefficients. We aim here to derive the universal behavior. For simplicity we set $n=1$. The corresponding state $|N-k;k\rangle$ in the FP system is a multiparticle state, built out of free harmonic oscillators, as in (2.19), containing $(N-k)$ quanta of negative angular momentum and k quanta of positive angular momentum. We will assume that the same picture holds in the D1-D5 system, at least in the large N limit, where the negative (positive) angular momenta quanta are now positive (negative) R-charge quanta.

We now treat $\mathcal{O}_{1,1}$ and $\mathcal{O}_{0,0}$ in similar way. $\mathcal{O}_{1,1}$ creates a quantum of positive R-charge and destroys a quantum of negative R-charge, so

$$\mathcal{O}_{1,1} \sim (a^{-12})^{\dagger}a^{+12}, \quad (9.13)$$

and $\mathcal{O}_{0,0}$ is the product of number operators for positive and negative R-charge quanta,

$$\mathcal{O}_{0,0} \sim \frac{1}{N}\left((a^{+12})^\dagger a^{+12}\right)\left((a^{-12})^\dagger a^{-12}\right), \tag{9.14}$$

where the normalization factor is introduced for later convenience.

Using standard harmonic oscillator relations then yields

$$\langle N-k; k|\mathcal{O}_{1,1}|N-k-1; k+1\rangle \sim \sqrt{(N-k)(k+1)}\mu^2, \tag{9.15}$$

with the scale μ^2 appropriate to a dimension two operator inserted, as in (8.11). Then the total vev for the ellipse is

$$(ellipse|\mathcal{O}_{1,1}|ellipse) \sim \sum_{k=0}^{N-1} \frac{\mu^2}{2^{2N}} \frac{N!}{(N-1-k)!k!}(a+b)^{2N-2k-1}(a-b)^{2k+1};$$
$$= \frac{N\mu^2}{2^{2N}}(2(a^2+b^2))^{N-1}(a^2-b^2) = \frac{1}{4}N\mu^2(a^2-b^2), \tag{9.16}$$

which indeed agrees in form with the vevs of charged operators in (7.16). The fact that such a simple approximation for the three point functions works so well merits further investigation.

For the neutral operators we obtain

$$\langle N-k; k|\mathcal{O}_{0,0}|N-k, k\rangle \sim \frac{1}{N}\mu^2(N-k)k, \tag{9.17}$$

and the corresponding total vev for this neutral operator is

$$(ellipse|\mathcal{O}_{0,0}|ellipse) \sim \sum_{k=1}^{N-1} \frac{\mu^2}{2^{2N}} \frac{(N-1)!}{(k-1)!(N-(1+k))!}(a+b)^{2(N-k)}(a-b)^{2k}; \tag{9.18}$$
$$= \frac{1}{2^{2N}}(N-1)\mu^2(a^2-b^2)^2(2(a^2+b^2))^{N-2} \sim \frac{1}{4}N\mu^2(1-a^2b^2),$$

in agreement with the vevs for uncharged operators given in (7.16). Note that (9.17) also gives zero for $k=0$ and $k=N$, in agreement with the vanishing vevs of the neutral operators for the circular case.

Now consider the more general ellipse of (7.17). The proposed dual in this case would be

$$|a,b,c,d\rangle = \sum_{k=0}^{N/n} \frac{1}{2^{\frac{N}{n}}} \sqrt{\frac{(\frac{N}{n})!}{(\frac{N}{n}-k)!k!}}(A_+)^{\frac{N}{n}-k}(A_-)^k (\mathcal{O}_n^{R++})^{(\frac{N}{n}-k)}(\mathcal{O}_n^{R--})^k,$$
$$A_\pm = (a\pm b) + i(c\mp d), \tag{9.19}$$

with $(a^2+b^2+c^2+d^2)=2$. Following the same steps as above, one finds exactly the R charges as in (7.18), supporting the proposal. As discussed below (7.18), charged operators $\mathcal{O}_{1,1}$ and $\mathcal{O}_{-1,-1}$ no longer have equal vevs. Repeating the steps which led to (9.10) and (9.11) one finds that

$$\left(\frac{A_+}{A_-}\right)^m \langle \mathcal{O}_{m,m}\rangle = \left(\frac{A_+^*}{A_-^*}\right)^m \langle \mathcal{O}_{-m,-m}\rangle. \tag{9.20}$$

Taking the case $m = 1$ this is indeed the relationship between the vevs $\langle \mathcal{O}_{\Sigma^2_{\pm 1, \pm 1}} \rangle$ and $\langle \mathcal{O}_{S^2_{\pm 1, \pm 1}} \rangle$ in (7.18), thus demonstrating that the proposal passes kinematical checks. Now let us compute the vevs of the dimension two charged operators using the same approximation (9.15) as before; this gives

$$\langle \mathcal{O}_{\pm 1, \pm 1} \rangle \sim N\mu^2((a \pm id)^2 + (c \pm ib)^2), \tag{9.21}$$

in agreement with (7.18). There is similar agreement for the behavior of the vevs of neutral operators $\mathcal{O}_{0,0}$. Of course, given the agreement for the ellipse above, there must be agreement for the rotated ellipse if the proposed dual captures correctly the orientation of the curve in the 1-2 plane. Nonetheless, this example clearly demonstrates how the parameters of the curve are captured by the (complex) coefficients in the linear superposition.

So to summarize: we have tested the proposed field theory dual in the case of elliptical curves. We find perfect agreement for all kinematically determined quantities, thus demonstrating the consistency of the proposal. We also find exact matching for the R charges and qualitative agreement for the vevs of the scalar operators. To test the correspondence further would require knowledge of three point functions involving multiparticle states at the conformal point.

10. Symmetric supergravity solutions

We next move to the question of whether one can find geometries which are dual to a single chiral primary, rather than a superposition of chiral primaries. As has already been discussed, a geometry which is dual to a chiral primary must preserve the $SO(2) \times SO(2)$ symmetry. This immediately implies that the asymptotics must be of the following form:

$$f_5 = \frac{Q_5}{r^2} \sum_{k=2l} \frac{f^5_{k0}}{r^k} Y^0_k; \tag{10.1}$$
$$f_1 = \frac{Q_1}{r^2} \sum_{k=2l} \frac{f^1_{k0}}{r^k} Y^0_k,$$

where the scalar spherical harmonics Y^0_{2l} which are singlets under $SO(2) \times SO(2)$ are defined in (A.19). The forms (A, B) must similarly admit an asymptotic expansion of the form:

$$A_a = \sum_k \frac{Q_5}{r^{k+1}} (A_{k0+} Y^{0+}_{ka} + A_{k0-} Y^{0-}_{ka}); \tag{10.2}$$
$$B_a = \sum_k \frac{Q_5}{r^{k+1}} (-A_{k0+} Y^{0+}_{ka} + A_{k0-} Y^{0-}_{ka}),$$

where the vector spherical harmonics $Y^{0\pm}_{ka}$ of degree k (k odd) whose Lie derivatives along the $SO(2)^2$ directions are zero are defined in (A.27). Note that these forms have only components along the (ϕ, ψ) directions. We will now give several examples of solutions which have asymptotics of this form, and discuss their interpretations.

10.1 Averaged geometries

Here we discuss a way to construct supergravity solutions based on a general closed curve F^i which are symmetric under $\mathrm{SO}(2) \times \mathrm{SO}(2)$ and thus have vanishing vevs for all charged operators. Let us first discuss the construction for arbitrary planar curves in the 1-2 plane. Starting from a general curve $(F^1, F^2, 0, 0)$ we construct a rotated curve,

$$\tilde{F}^1 = \cos\alpha F^1 + \sin\alpha F^2, \qquad \tilde{F}^2 = -\sin\alpha F^1 + \cos\alpha F^2, \tag{10.3}$$

and then superimpose the solutions. This leads to a new harmonic function,

$$f_5 = \int_0^{2\pi} \frac{d\alpha}{2\pi} \frac{Q_5}{L} \int_0^L \frac{dv}{|x - \tilde{F}|^2} = \frac{Q_5}{L} \int_0^L \frac{dv}{\sqrt{(r^2 + |F|^2)^2 - 4r^2|F|^2 \sin^2\theta}} \tag{10.4}$$

where we use coordinates on R^4 such that $(x^1)^2 + (x^2)^2 = r^2 \sin^2\theta$, $(x^3)^2 + (x^4)^2 = r^2 \cos^2\theta$. The harmonic function for f_1 is the same as f_5 in (10.4) but with the numerator on the rhs multiplied by $|\dot{F}|^2$. The non-vanishing part of the gauge field is given by

$$A_\phi = \frac{Q_5}{L} \int_0^L \frac{\dot{F}^{[1}F^{2]}dv}{|F|^2} \left(1 - \frac{r^2 + |F|^2}{\sqrt{(r^2 + |F|^2)^2 - 4r^2|F|^2 \sin^2\theta}}\right), \tag{10.5}$$

where ϕ is a polar coordinate in the 1-2 plane and square brackets indicate antisymmetrization with unit strength. The only non-vanishing component of the dual form B is

$$B_\psi = \frac{Q_5}{L} \int_0^L \frac{\dot{F}^{[1}F^{2]}dv}{|F|^2} \left(\frac{r^2 - |F|^2}{\sqrt{(r^2 + |F|^2)^2 - 4r^2|F|^2 \sin^2\theta}} - 1\right). \tag{10.6}$$

where ψ is a polar coordinate in the 3-4 plane. For a general curve (F^1, F^2, F^3, F^4) we can proceed analogously by considering solutions rotated by angle α in the 1-2 plane and by angle β in 3-4 plane and then averaging over α and β. For example, the function f_5 would be given by

$$f_5 = \int_0^{2\pi} \frac{d\beta}{2\pi} \frac{Q_5}{L} \int_0^L \frac{dv}{\sqrt{(r^2 + |F|^2 - 2r\cos\theta g(\beta))^2 - 4r^2((F^1)^2 + (F^2)^2)\sin^2\theta}};$$
$$g(\beta) = (F^3 \cos(\psi + \beta) + F^4 \sin(\psi + \beta)). \tag{10.7}$$

This integral can be expressed in terms of elliptic integrals, although we have not obtained the exact form. The asymptotics are however given by:

$$f_5 = \frac{Q_5}{Lr^2} \int_0^L \sum_{l \geq 0} \frac{dv}{r^{2l}} P_l(\cos(2\theta)) P_l(Z(F(v)));$$

$$f_1 = \frac{Q_5}{Lr^2} \int_0^L \sum_{l \geq 0} \frac{dv}{r^{2l}} |\partial_v F|^2 P_l(\cos(2\theta)) P_l(Z(F(v))); \tag{10.8}$$

$$A = \frac{Q_5}{L} \int_0^L \sum_k \frac{dv}{\sqrt{2}(k+1)r^{k+1}} \left(p_k(F)(\dot{F}^1 F^2 - \dot{F}^2 F^1)(Y_{ka}^{0+} - Y_{ka}^{0-})\right.$$
$$\left. + q_k(F)(\dot{F}^3 F^3 - \dot{F}^4 F^3)(Y_{ka}^{0+} + Y_{ka}^{0-})\right);$$

$$Z(F) = (F^3)^2 + (F^4)^2 - (F^1)^2 - (F^2)^2,$$

where $P_l(x)$ are Legendre polynomials of degree l and $p_k(F)$ and $q_k(F)$ are defined in (A.29)–(A.31). These asymptotics are manifestly of the form given in (10.1) and (10.2). Setting $F^3 = F^4 = 0$ gives the asymptotic expansion of the expressions given in (10.4) and (10.5).

10.1.1 Example 1: the averaged ellipse

Consider the case of an ellipse, so that the defining curve is

$$F^1 = \mu a \cos \frac{2\pi v}{L}, \quad F^2 = \mu b \sin \frac{2\pi v}{L}, \tag{10.9}$$

with $\mu = \sqrt{Q_1 Q_5}/R$ and $(a^2 + b^2) = 2$. (For simplicity we choose the frequency n to be one.) In this example the integral over the curve in (10.4) can be carried out explicitly to give

$$f_5 = \frac{2Q_5}{\pi z} K(w); \tag{10.10}$$

$$z^4 = (C^2 + D^2); \quad w = \frac{\sqrt{(z^2 - C)}}{\sqrt{2} z};$$

$$C = (r^4 + 2\mu^2 r^2 \cos 2\theta + \mu^4 a^2 b^2);$$

$$D = \mu^2 r^2 \sin 2\theta (a^2 - b^2),$$

where $K(w)$ is the complete elliptic integral of the first kind. Then f_5 has poles only where z has zeroes, namely at $\theta = \pi/2$ and $r = \mu a$ or $r = \mu b$. This suggests that any singularities of the metric are confined to these locations, namely circles of radius a and b in the 1-2 plane, and indeed one finds that the other defining functions (f_1, A, B) only have poles at these locations. Thus the geometry is less singular than one might have anticipated. The integrands have singularities in the annular region defined by $\theta = \pi/2$ and $\mu a \leq r \leq \mu b$ (assuming $a \leq b$) but the integrated functions only have singularities on the circles bounding this annulus. Moreover these singularities seem to be such that the only singularities of the resulting metric are conical.

10.1.2 Example 2: Aichelburg-Sexl metric

The Aichelburg-Sexl metric was also obtained by the procedure of averaging over curve orientations in [2]. The defining curve has a section which is constant:

$$F^1 = a \cos(\frac{2\pi v}{\xi L}); \quad F^2 = a \sin(\frac{2\pi v}{\xi L}), \quad 0 \leq v \leq \xi L; \tag{10.11}$$

$$F^1 = a, \quad \xi L \leq v \leq L,$$

with all other $F^i(v) = 0$ and $\xi < 1$. Such a profile gives rise to the following harmonic functions:

$$f_5 = \left(1 + \frac{Q_5 \xi}{\hat{r}^2 + a^2 \cos^2 \hat{\theta}} + \frac{Q_5(1 - \xi)}{(x_1 - a)^2 + x_2^2 + x_3^2 + x_4^2}\right); \tag{10.12}$$

$$f_1 = \left(1 + \frac{Q_1}{\hat{r}^2 + a^2 \cos^2 \hat{\theta}}\right);$$

$$A_\phi = a\sqrt{\xi} \frac{\sqrt{Q_1 Q_5}}{\hat{r}^2 + a^2 \cos^2 \hat{\theta}},$$

where $Q_1 = Q_5 a^2 (2\pi/L)^2/\xi$ and as in (7.5) we introduce non-standard polar coordinates on R^4 to simplify the harmonic functions. Now we take the SO(2) orbit of the defining curve, thus averaging over the location of the constant section in the 1-2 plane. This leads to the SO(2) symmetric harmonic functions

$$f_5 = \left(1 + \frac{Q_5}{\hat{r}^2 + \xi \mu^2 \cos^2 \hat{\theta}}\right); \tag{10.13}$$

$$f_1 = \left(1 + \frac{Q_1}{\hat{r}^2 + \xi \mu^2 \cos^2 \hat{\theta}}\right);$$

$$A_\phi = \frac{\xi \mu \sqrt{Q_1 Q_5}}{\hat{r}^2 + \xi \mu^2 \cos^2 \hat{\theta}},$$

which are those of the Aichelburg-Sexl metric

$$ds^2 = f_1^{-1/2} f_5^{-1/2} \left(-\left(dt - \frac{\xi \mu \sqrt{Q_1 Q_5}}{\hat{r}^2 + \xi \mu^2 \cos^2 \hat{\theta}} \sin^2 \hat{\theta} d\phi\right)^2 + \left(dy - \frac{\xi \mu \sqrt{Q_1 Q_5}}{\hat{r}^2 + \xi \mu^2 \cos^2 \hat{\theta}} \cos^2 \hat{\theta} d\psi\right)^2\right)$$

$$+ f_1^{1/2} f_5^{1/2} \left((\hat{r}^2 + \xi \mu^2 \cos^2 \hat{\theta})\left(\frac{d\hat{r}^2}{\hat{r}^2 + \xi \mu^2} + d\hat{\theta}^2\right) + \hat{r}^2 \cos^2 \hat{\theta} d\psi^2 + (\hat{r}^2 + \xi \mu^2) \sin^2 \hat{\theta} d\phi^2\right).$$

Here $\mu = \sqrt{Q_1 Q_5}/R$. This solution is clearly very similar to those based on circular curves, discussed in section 7.1. The non-zero vevs extracted from the decoupled part of the geometry follow from (6.4) and are given by

$$\langle J^{+3} \rangle = \frac{N}{4\pi} \mu \xi (dy - dt); \quad \langle J^{-3} \rangle = \frac{N}{4\pi} \mu \xi (dy + dt); \tag{10.14}$$

$$\left\langle \mathcal{O}_{\Sigma_0^2} \right\rangle = \frac{N\sqrt{2}\xi\mu^2}{2\pi\sqrt{3}}(1 - \xi).$$

These clearly reduce to those for the case of the circular curves when $\xi = 1$. Note that the Aichelburg-Sexl metrics do not have conical singularities, and are therefore actually less singular than the unaveraged geometries. However, whilst the Aichelburg-Sexl metrics do have the correct asymptotics to correspond to chiral primaries, they are based on averaging curves with straight sections. The interpretation of these straight sections from the dual perspective is rather unclear, given the proposed correspondence between frequencies on the curve and twists of the dual operators.

10.2 Disconnected curves

Another way to obtain solutions which preserve the $SO(2) \times SO(2)$ symmetry is to consider curves made up of disconnected circles. There exist supergravity solutions defined by the following functions

$$f_5 = \sum_{l=1}^{I} \frac{Q_5 N_l}{NL} \int_0^L \frac{dv_l}{|x - F_l|^2}; \tag{10.15}$$

$$f_1 = \sum_{l=1}^{I} \frac{Q_5 N_l}{NL} \int_0^{L_l} \frac{dv_l (\partial_{v_l} F_l)^2}{|x - F_l|^2};$$

$$A_i = \sum_{l=1}^{I} \frac{Q_5 N_l}{NL} \int_0^{L_l} \frac{dv_l \partial_{v_l} F_l^i}{|x - F_l|^2},$$

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where the lth curve is parametrized by v_l with $\sum_l N_l = N$ and is circular within either the 1-2 or 3-4 plane. That is, the curve defining the lth circle is given by

$$F_l^1 = \frac{\sqrt{Q_1 Q_5}}{R n_l} \cos\left(\frac{2\pi n_l v_l}{L}\right); \quad F_l^2 = \pm \frac{\sqrt{Q_1 Q_5}}{R n_l} \sin\left(\frac{2\pi n_l v_l}{L}\right), \quad (10.16)$$

assuming the circle lies in the 1-2 plane; the sign determines the direction of rotation. A curve lying in the 3-4 plane will take an analogous form. Such a linear superposition of sources solves the field equations and is supersymmetric. By construction the total D5-brane and D1-brane charges are Q_5 and Q_1 respectively, with the lth curve sourcing a fraction N_l/N of (both) the total charges. The related radii and frequencies in (10.16) ensure that the D1-brane charge of each curve is a fraction N_l/N of the total. This prescription also reduces to that given for the curves corresponding to the operators (9.1); in that case one lets $I = N/n$ and $N_l = n$ in the supergravity solution above and takes the circles to be coincident. Furthermore the total R-charges will be given by

$$j_3 = \frac{1}{2} \sum_{l=1}^{I} \epsilon_l m_l; \quad \bar{j}_3 = \frac{1}{2} \sum_{l=1}^{I} \bar{\epsilon}_l m_l, \quad (10.17)$$

where $m_l = N_l/n_l$. Here $(\epsilon_l, \bar{\epsilon}_l) = (\pm 1, \pm 1)$ depending on the orientation and rotation of the curve.

Since the sources are located on disconnected circles, the singularity structure of these geometries is similar to that discussed in section 7.1. Namely, there are conical singularities whenever $n_l \neq 1$. Thus, these solutions are no more singular than the geometries based on a single circle, although they are more singular than a geometry based on a general non-intersecting curve.

10.3 Discussion

These are not the only symmetric geometries. For example, one could consider more general superpositions of curves, superposing not just different orientation curves but also different shape curves. However, the procedure we outlined above does illustrate how symmetric geometries can be obtained from those defined in terms of a single curve. The symmetrization we used is the simplest, in that the measure for each curve is the same. The field theory dual suggests that symmetrizing over shapes of curves should involve a non-trivial measure. That is, if one has an ellipse with parameters (a, b) so that the proposed dual is

$$|ellipse\rangle_{a,b} = \sum_{k=0}^{N/n} (a_k)_{a,b} |\frac{N}{n} - k; k\rangle, \quad (10.18)$$

then one can formally invert the relation to give

$$|\frac{N}{n} - k; k\rangle = \sum_{a,b} (a_k)_{a,b}^{-1} |ellipse\rangle_{a,b}. \quad (10.19)$$

This suggests that to obtain a geometric dual for a given chiral primary one could consider a linear superposition of curves with different parameters (a, b) using a measure which

is related to $(a_k)_{a,b}^{-1}$. Precisely what the measure should be is not however immediately apparent, because, as we will discuss below, such a symmetrization via linear superposition may in fact be rather too naive, because of the non-linear relationship between harmonic functions and vevs. To test whether a given symmetric geometry does indeed have the correct properties to correspond to a given chiral primary, one will need to use the actual values of the kinematically allowed vevs, as we will now discuss.

11. Dynamical tests for symmetric geometries

The geometries in sections 10.1 and 10.2 have the correct asymptotics to correspond to chiral primaries. Since the geometries in section 10.2 are based on separated sources, one would not however anticipate that these correspond to Higgs branch vacua; the more natural proposal would be that they relate to Coulomb branch vacua. By extracting all vevs and n-point functions from each geometry one could in principle identify the field theory dual uniquely.

Furthermore, given any proposed correspondence between geometries and field theory vacua, we can use dynamical information for the kinematically allowed vevs to test it. In particular, let us consider the averaged geometries, focusing on the example of the averaged ellipse. In this case, we consider a defining curve (10.9) with corresponding rotated curve $\tilde{F}^1, \tilde{F}^2$ defined in (10.3). The geometry based on the latter is proposed to correspond to the linear superposition (9.19) with

$$A_+ = (a + b)e^{i\alpha}; \quad A_- = (a - b)e^{-i\alpha}. \tag{11.1}$$

This means that the superposition dual to the rotated ellipse is

$$|ellipse)_\alpha = \sum_{k=0}^{N/n} e^{i\alpha(\frac{N}{n} - 2k)} \frac{1}{2^{\frac{N}{n}}} \sqrt{\frac{(\frac{N}{n})!}{(\frac{N}{n} - k)!k!}} (a+b)^{\frac{N}{n} - k}(a-b)^k (\mathcal{O}_n^{R++})^{(\frac{N}{n} - k)}(\mathcal{O}_n^{R--})^k. \tag{11.2}$$

Averaging over the angle α clearly picks out the $k = N/2n$ term in the superposition, which is a state of zero angular momentum. However, the geometry obtained by averaging over rotated ellipses does not have zero angular momentum, but rather the same angular momentum as the original geometry. This suggests that this geometric averaging might actually average over vevs, rather than over states, and thus not pick out a geometry dual to a single chiral primary. Given that the averaging linearly superposes harmonic functions, however, and the vevs are non-linearly related to the harmonic functions, the geometric averaging probably does not lead to just an overall averaging over the vevs. One will have to use the actual vevs for the neutral operators to see what the geometry describes.

So now let us discuss how one would use information about three point functions at the conformal point to test whether a given geometry corresponds to a chiral primary. Let us work with an example: consider the R vacuum corresponding to the operator $(\mathcal{O}_{S_n^p})_R$ obtained by spectrally flowing the operator $\mathcal{O}_{S_n^p}$ dual to the supergravity field $S_n^{p(6)}$. (Recall that the superscript p denotes that it is primary, $j^3 = j$ and $\bar{j}^3 = \bar{j}$.) Next suppose that there is a candidate dual geometry, which has the correct symmetries and R-charges, the

latter being $(\frac{1}{2}(n-N), \frac{1}{2}(n-N))$. This means that the holographic vevs for the R symmetry currents must be

$$\langle J^{\pm 3}\rangle = \frac{\mu}{4\pi}(n - N)(dy \mp dt), \tag{11.3}$$

where y has periodicity $2\pi\tilde{R} = 2\pi R/\sqrt{Q_1 Q_5}$ and $\mu = \sqrt{Q_1 Q_5}/R$.

Now let us consider how we can relate this vev to the normalized three point function at the conformal point. That is,

$$\langle J^{\pm 3}\rangle_{\Psi_{S_n}} = \langle (\mathcal{O}_{S^n})_R^\dagger J^{\pm 3}(w_0)(\mathcal{O}_{S^n})_R\rangle \equiv \frac{\langle (\mathcal{O}_{S^n})_R^\dagger(\infty) J^{\pm 3}(w_0)(\mathcal{O}_{S^k})_R(0)\rangle}{\langle (\mathcal{O}_{S^n})_R^\dagger(\infty)(\mathcal{O}_{S^n})_R(0)\rangle}, \tag{11.4}$$

where Ψ_{S_n} denotes that the theory is in the vacuum created by $(\mathcal{O}_{S_n^p})_R$. The scale w_0 at which the current is inserted is found by comparing the vevs (11.3) with the normalized three point functions, computed in (D.25). The latter give

$$\langle J^{+3}\rangle_{\Psi_{S_n}} = \frac{(n-N)}{4\pi w_0}; \qquad \langle J^{-3}\rangle_{\Psi_{S_n}} = \frac{(n-N)}{4\pi \bar{w}_0}, \tag{11.5}$$

which comparing with (11.3) implies that the inserted scale must be $w_0 = \bar{w}_0 = \mu^{-1}$.

We can now use the three point functions between $\mathcal{O}_{S_n^p}$ and neutral dimension two operators to predict the vevs for the latter. This gives

$$\langle \mathcal{O}_{S_0^2}\rangle_{\Psi_{S_n}} = 0; \tag{11.6}$$

$$\langle \mathcal{O}_{\Sigma_0^2}\rangle_{\Psi_{S_n}} = \langle (\mathcal{O}_{S^n})_R^\dagger \mathcal{O}_{\Sigma_0^2}(\mu^{-1})(\mathcal{O}_{S^n})_R\rangle = \frac{\sqrt{3}n^3\mu^2}{\sqrt{2}\pi(n-1)^2}.$$

where the normalized three point function is defined in (D.18) and the inserted scale is as before $w_0 = \bar{w}_0 = \mu^{-1}$. Note that $\mu^2 \sim N$, so the vev has the correct large N behavior (for our choice of normalization). From the expressions given in (6.4) for the vevs of these operators in terms of the asymptotics we can determine the degree two coefficients in (10.1). The vanishing of $\langle \mathcal{O}_{S_0^2}\rangle_{\Psi_{S_n}}$ implies that $f_{20}^1 = f_{20}^5$ whilst the expression for the vev $\langle \mathcal{O}_{\Sigma_0^2}\rangle_{\Psi_{S_n}}$ in (6.4) implies that

$$f_{20}^1 = -\frac{\mu^2}{\sqrt{3}N^2}\left((n-N)^2 + \frac{3n^3 N}{(n-1)^2}\right) \tag{11.7}$$

$$= f_{20}^1(circ)\left(1 + \frac{n}{N} + \cdots\right),$$

where $f_{20}^1(circ) = -\mu^2/\sqrt{3}$ is the value of f_{20}^1 for the circular solution. The $(n - N)^2$ contribution on the rhs is due to the non-linear contribution $8a^{\alpha-}a^{\beta+}f_{I\alpha\beta}$ and in the second equality we use $1 \ll n \ll N$. The upper limit on n follows from the fact that the supergravity three point functions are known only to leading order in N and do not apply for operators with dimensions comparable to N. The lower limit is unnecessary and is imposed only to simplify the formula.

By extending the computation of the vevs to higher dimension operators and comparing with those predicted from three point functions at the conformal point, one could in

principle extract the higher degree coefficients in (10.1) and resum the asymptotic series to obtain the full geometry.

There is an important caveat, however. In all computations so far we have worked in the $N \to \infty$ limit, retaining only the leading terms. This applies both to the computation of the vevs and to the computation of three point functions. For the computation of the 3-point function to be valid we need $N \gg n$, but then the "holographically engineered" f_{20}^1 in (11.7) differs from the answer for the circle only by terms subleading in n/N. In other words, the holographically engineered geometry would be that of the circular solution up to $1/N$ corrections.

Next consider R vacua corresponding to operators obtaining by spectral flow on operators which are either of high dimension (comparable to N) or multiparticle. The latter include operators of the form $(\mathcal{O}_n^{R++})^{N/n-k}(\mathcal{O}_n^{R--})^k$ for which the duals may be related to averaged ellipses. Since there is no information about three point functions of these operators at strong coupling, we have no precise predictions for the vevs of neutral operators and thus cannot currently test whether a given geometry is indeed dual to such a state. Given any future progress on computing the relevant fusion coefficients via string theory, one could however test the correspondence further.

To summarize: a geometry with $SO(2) \times SO(2)$ symmetry can be characterized by its angular momentum and vevs of neutral operators. The latter can in principle be used to determine the corresponding dual, but to implement this program will in general require going beyond the leading supergravity approximation.

12. Including the asymptotically flat region

In this section we will discuss how the asymptotically flat region of the geometry may be interpreted using the AdS/CFT dictionary. Our discussion will parallel an analogous discussion for D3-branes given in section 6 of [35].

The six-dimensional metric of (3.8) along with the scalar and tensor field of (3.1) are characterized by two harmonic functions (f_1, f_5) and a harmonic form A_i. The field equations are satisfied for any choice of harmonic functions. The specific choices in (3.4) correspond to (part of) the (supersymmetric) Higgs branch of the D1-D5 system. Multi-centered harmonic functions for (f_1, f_5) with $A_i = 0$ are also well-known supergravity solutions, corresponding to part of the Coulomb branch.

In (4.3) we gave the most general form for the asymptotic expansions of (f_1, f_5, A_i) under the condition that the solution is asymptotically $AdS_3 \times S^3$. The asymptotically flat region may be included by adding constant terms to the (f_1, f_5) harmonic expansions, namely

$$f_1 = \epsilon_1 + \frac{Q_1}{r^2} \sum_{k,I} \frac{f_{kI}^1 Y_k^I(\theta_3)}{r^k}; \quad f_5 = \epsilon_5 + \frac{Q_5}{r^2} \sum_{k,I} \frac{f_{kI}^5 Y_k^I(\theta_3)}{r^k}, \tag{12.1}$$

whilst keeping the large radius expansion for A_i as in (4.3). To include all of the asymptotically flat region, the parameters ϵ_1 and ϵ_5 clearly need to be finite. However, let us take the parameters to be infinitesimal so that the solution remains asymptotically $AdS_3 \times S^3$.

Since we have discussed already the terms in the harmonic expansion behaving as r^{-k} with $k \geq 3$, we consider only the new terms as a perturbation to the AdS background. That is, we let

$$f_1 = \epsilon_1 + \frac{Q_1}{r^2}; \quad f_5 = \epsilon_5 + \frac{Q_5}{r^2}, \tag{12.2}$$

with $A_i = 0$ and then identify the terms induced in the harmonic expansion of the fluctuations (4.2). The field fluctuations are

$$-h_{tt} = h_{yy} = -\frac{1}{2}r^4(\hat{\epsilon}^1 + \hat{\epsilon}^5); \quad h_{rr} = \frac{1}{2}(\hat{\epsilon}^1 + \hat{\epsilon}^5); \tag{12.3}$$

$$h_{ab} = \frac{1}{2}r^2(\hat{\epsilon}^1 + \hat{\epsilon}^5); \quad \phi^{(56)} = \frac{1}{2}r^2(\hat{\epsilon}^1 - \hat{\epsilon}^5);$$

$$g^5_{tyr} = -r^3(\hat{\epsilon}^1 + \hat{\epsilon}^5); \quad g^6_{tyr} = -r^3(\hat{\epsilon}^1 - \hat{\epsilon}^5),$$

where we define $\hat{\epsilon}^1 = \epsilon^1/Q_1$ and $\hat{\epsilon}^5 = \epsilon^5/Q_5$. Thus the only non-vanishing dynamical fields are those from (5.1)

$$\tau_0 \equiv \frac{\pi^0}{12} = \frac{1}{8}r^2(\hat{\epsilon}^1 + \hat{\epsilon}^5); \quad t_0 \equiv \frac{1}{4}\phi_0^{(56)} = \frac{1}{8}r^2(\hat{\epsilon}^1 - \hat{\epsilon}^5). \tag{12.4}$$

(The other non-vanishing components are induced by constraint equations and do not correspond to dynamical fields.) Since both τ_0 and t_0 couple respectively to the dimension four operators $\mathcal{O}_{\tau_0}$ and $\mathcal{O}_{t_0}$, the radial dependence of these fields corresponds to source behavior. Thus the CFT lagrangian is deformed by the terms

$$\int d^2z \left((\hat{\epsilon}^1 + \hat{\epsilon}^5)\mathcal{O}_{\tau_0} + (\hat{\epsilon}^1 - \hat{\epsilon}^5)\mathcal{O}_{t_0} \right). \tag{12.5}$$

Note that the operators $(\mathcal{O}_{\tau_0}, \mathcal{O}_{t_0})$ are the top components of the short multiplets generated from the chiral primaries $(\mathcal{O}_{\Sigma_2}, \mathcal{O}_{S_2})$ respectively through the action of the supercharges. That is, they are given by

$$G^{1\dagger}_{-1/2}G^2_{-1/2}\tilde{G}^{1\dagger}_{-1/2}\tilde{G}^2_{-1/2} |CPO\rangle, \tag{12.6}$$

where $(G^a_{\pm1/2}, \tilde{G}^a_{\pm1/2})$ with $a = 1, 2$ are left and right supercharges. Here $(G^{1\dagger}_{-1/2}, G^2_{-1/2})$ and corresponding right moving charges act as raising operators on the $\Delta = 2$ chiral primaries. The latter have $h = j = j^3 = \bar{h} = \bar{j} = \bar{j}^3 = 1$. Computing two point functions in the presence of the deformation (12.5) may capture scattering into the asymptotically flat part of the D1-D5 geometry.

Acknowledgments

A. Properties of spherical harmonics

Scalar, vector and tensor spherical harmonics satisfy the following equations

$$\Box Y^I = -\Lambda_k Y^I, \tag{A.1}$$
$$\Box Y_a^{I_v} = (1 - \Lambda_k)Y_a^{I_v}, \qquad D^a Y_a^{I_v} = 0,$$
$$\Box Y_{(ab)}^{I_t} = (2 - \Lambda_k)Y_{(ab)}^{I_t}, \qquad D^a Y_{k(ab)}^{I_t} = 0,$$

where $\Lambda_k = k(k+2)$ and the tensor harmonic is traceless. It will often be useful to explicitly indicate the degree k of the harmonic; we will do this by an additional subscript k, e.g. degree k spherical harmonics will also be denoted by Y_k^I, etc. $\Box$ denotes the d'Alambertian along the three sphere. The vector spherical harmonics are the direct sum of two irreducible representations of $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ which are characterized by

$$\epsilon_{abc}D^b Y^{cI_v\pm} = \pm(k+1)Y_a^{I_v\pm} \equiv \lambda_k Y_a^{I_v\pm}. \tag{A.2}$$

The degeneracy of the degree k representation is

$$d_{k,\epsilon} = (k+1)^2 - \epsilon, \tag{A.3}$$

where $\epsilon = 0, 1, 2$ respectively for scalar, vector and tensor harmonics. For degree one vector harmonics I_v is an adjoint index of $\mathrm{SU}(2)$ and will be denoted by α.

We use normalized spherical harmonics such that

$$\int Y^{I_1}Y^{J_1} = \Omega_3 \delta^{I_1 J_1}; \quad \int Y^{aI_v}Y_a^{J_v} = \Omega_3 \delta^{I_v J_v}; \quad \int Y^{(ab)I_t}Y_{(ab)}^{J_t} = \Omega_3 \delta^{I_t J_t}, \tag{A.4}$$

where $\Omega_3 = 2\pi^2$ is the volume of a unit 3-sphere. Then

$$\int D_a Y^{I_1} D^a Y^{J_1} = \Omega_3 \Lambda^{I_1} \delta^{I_1 J_1}; \quad \int D^{(a}D^{b)}Y^{I_1}D_a D_b Y^{I_2} = \Omega_3 \frac{2}{3}\Lambda^{I_1}(\Lambda^{I_1} - 3)\delta^{I_1 J_1}. \tag{A.5}$$

The following identities are useful

$$\frac{1}{\Omega_3}\int Y^I D^a Y^J D_a Y^K \equiv b_{IJK} = \frac{1}{2}(\Lambda^J + \Lambda^K - \Lambda^I)a_{IJK}; \tag{A.6}$$

$$\frac{1}{\Omega_3}\int D^{(a}D^{b)}Y^I D_a D_b Y^J Y^K \equiv c_{IJK} = \left(\frac{1}{4}\Lambda_{IJK}(\Lambda_{IJK} - 4) - \frac{1}{3}\Lambda^I \Lambda^J\right)a_{IJK};$$

$$\frac{1}{\Omega_3}\int D_{(a}Y^I D_{b)}Y^J D^a D^b Y^K \equiv d_{IJK} = \left(\frac{1}{4}\Lambda_{IKJ}\Lambda_{JKI} + \frac{1}{6}\Lambda^K \Lambda_{IJK}\right)a_{IJK},$$

where $\Lambda_{IJK} = (\Lambda^I + \Lambda^J - \Lambda^K)$. We define the following triple integrals as

$$\int Y^I Y^J Y^K = \Omega_3 a_{IJK}; \tag{A.7}$$

$$\int (Y_1^{\alpha\pm})^a Y_1^j D_a Y_1^i = \Omega_3 e_{\alpha ij}^{\pm}; \tag{A.8}$$

$$\int Y^I (Y_1^{\alpha-})_a (Y_1^{\beta+})^a = \Omega_3 f_{I\alpha\beta}; \tag{A.9}$$

$$\int (Y_{k_v}^{I_v\pm})^a Y_k^I D_a Y_1^i = \Omega_3 E_{I_v Ii}^{\pm}; \tag{A.10}$$

$$\int (Y_{k_v}^{I_v\pm})^a Y_k^I D_a Y_l^J = \Omega_3 E_{I_v IJ}^{\pm}; \tag{A.11}$$

We also use specific identities for harmonics of low degree. The degree one vector harmonics $Y_{1\pm}^{\alpha}$ transform in the $(1,0)$ and $(0,1)$ representation of $(SU(2)_L, SU(2)_R)$ whilst the degree k scalar harmonics transform in the $(\frac{1}{2}k, \frac{1}{2}k)$ representation. This immediately implies that the following triple overlaps are zero:

$$\int Y_2^I (Y_1^{\alpha+})_a (Y_1^{\beta+})^a) = \int Y_2^I (Y_1^{\alpha-})_a (Y_1^{\beta-})^a) = \int Y_0 (Y_1^{\alpha+})_a (Y_1^{\beta-})^a) = 0. \qquad (A.12)$$

Using the following explicit coordinate system on the sphere

$$ds_3^2 = d\theta^2 + \sin^2\theta d\phi^2 + \cos^2\theta d\psi^2, \qquad (A.13)$$

with volume form $\eta_3 = \sin\theta \cos\theta d\theta \wedge d\phi \wedge d\psi$ the following are normalized Killing forms

$$Y_1^{3+} = (\sin^2\theta d\phi + \cos^2\theta d\psi); \quad Y_1^{3-} = -(\sin^2\theta d\phi - \cos^2\theta d\psi), \qquad (A.14)$$

which generate the Cartan of the SO(4) symmetry group. The remaining Killing forms are

$$\begin{aligned}
Y_1^{1+} &= (\cos(\psi+\phi)d\theta + \sin(\psi+\phi)\sin\theta\cos\theta d(\psi-\phi)); \\
Y_1^{2+} &= (-\sin(\psi+\phi)d\theta + \cos(\psi+\phi)\sin\theta\cos\theta d(\psi-\phi)); \\
Y_1^{1-} &= (\cos(\psi-\phi)d\theta + \sin(\psi-\phi)\sin\theta\cos\theta d(\phi+\psi)); \\
Y_1^{2-} &= (-\sin(\psi-\phi)d\theta + \cos(\psi-\phi)\sin\theta\cos\theta d(\phi+\psi)).
\end{aligned}$$

The $SU(2) \times SU(2)$ algebra realized by the Killing vectors is normalized such that

$$[Y_1^{\alpha+}, Y_1^{\beta+}] = 2\epsilon_{\alpha\beta\gamma} Y_1^{\gamma+}; \quad [Y_1^{\alpha-}, Y_1^{\beta-}] = 2\epsilon_{\alpha\beta\gamma} Y_1^{\gamma-}; \quad [Y_1^{\alpha+}, Y_1^{\beta-}] = 0. \qquad (A.15)$$

Furthermore

$$Y_1^{\alpha\pm} \wedge Y_1^{\beta\pm} \wedge Y_1^{\gamma\pm} = \mp \epsilon_{\alpha\beta\gamma} \eta_3, \qquad (A.16)$$

which implies that

$$\int \epsilon^{abc} Y_a^{\alpha\pm} Y_b^{\beta\pm} Y_c^{\gamma\pm} = \mp \Omega_3 \epsilon_{\alpha\beta\gamma} \qquad (A.17)$$

In the same coordinate system $Y_2^0 = \sqrt{3}\cos 2\theta$ is the normalized degree 2 spherical harmonic which is a singlet under the $SO(2)^2$ Cartan, with the following triple overlap

$$\int Y_2^0 (Y_1^{3+})^a (Y_1^{3-})_a = \frac{1}{\sqrt{3}} \Omega_3. \qquad (A.18)$$

Thus, $f_{033} = 1/\sqrt{3}$ in this specific case. More generally the normalized spherical harmonics which are singlets under the Cartan can be expressed as

$$Y_{2l}^0 = \sqrt{2l+1} P_l(\cos 2\theta), \qquad (A.19)$$

where $P_l(x)$ is a Legendre polynomial of degree l, normalized so that $P_l(1) = 1$ and $P_l(-1) = (-1)^l$.

In this coordinate system normalized degree one spherical harmonics are

$$\begin{aligned}
Y_1^1 &= 2\sin\theta\cos\phi; \quad & Y_1^2 &= 2\sin\theta\sin\phi; \\
Y_1^3 &= 2\cos\theta\cos\psi; \quad & Y_1^4 &= 2\cos\theta\sin\psi.
\end{aligned} \qquad (A.20)$$

Defining $Y^{ij} \equiv \frac{1}{2}(Y_1^j dY_1^1 - Y_1^1 dY_1^j)$,

$$Y^{12} = (Y_1^{3-} - Y_1^{3+}); \quad Y^{34} = -(Y_1^{3+} + Y_1^{3-}); \quad Y^{13} = (Y_1^{1+} + Y_1^{1-}); \quad \text{(A.21)}$$
$$Y^{34} = (Y_1^{1-} - Y_1^{1+}); \quad Y^{14} = -(Y_1^{2+} + Y_1^{2-}); \quad Y^{23} = (Y_1^{2+} - Y_1^{2-}),$$

and therefore the explicit values for the overlaps $e_{ij}^{\pm\alpha}$ defined in (A.8) are

$$e_{12}^{+3} = -1; \quad e_{12}^{-3} = 1; \quad e_{34}^{+3} = -1; \quad e_{34}^{-3} = -1; \quad e_{13}^{+1} = 1; \quad e_{13}^{-1} = 1; \quad \text{(A.22)}$$
$$e_{24}^{+1} = -1; \quad e_{24}^{-1} = 1; \quad e_{14}^{+2} = -1; \quad e_{14}^{-2} = -1; \quad e_{23}^{+2} = 1; \quad e_{23}^{-2} = -1.$$

Note that $e_{ij}^{\pm\alpha} = -e_{ji}^{\pm\alpha}$.

We will also make use of normalized degree k scalar harmonics with maximal $(m,\bar{m})$ $(\mathrm{SU}(2)_L, \mathrm{SU}(2)_R)$ charges:

$$Y_k^{\pm\frac{1}{2}k,\pm\frac{1}{2}k} = \sqrt{k+1}\sin^k\theta e^{\pm ik\phi}; \tag{A.23}$$
$$Y_k^{\pm\frac{1}{2}k,\mp\frac{1}{2}k} = \sqrt{k+1}\cos^k\theta e^{\pm ik\psi}.$$

The triple overlap between two such harmonics of opposite charges with the neutral harmonic of degree two given in (A.19) is given by

$$\frac{1}{2\pi^2}\int Y_k^{\frac{1}{2}k,\frac{1}{2}k}Y_k^{-\frac{1}{2}k,-\frac{1}{2}k}Y_2^0 = -\frac{\sqrt{3}k}{k+2}. \tag{A.24}$$

We will also need the explicit values of the overlaps between two such harmonics of opposite charges and the commuting Killing vectors:

$$E_{3(--)(++)}^{\pm} \equiv \frac{1}{2\pi^2}\int D^a Y_k^{\frac{1}{2}k,\frac{1}{2}k}Y_k^{-\frac{1}{2}k,-\frac{1}{2}k}Y_a^{3\pm} = \pm ik; \tag{A.25}$$

$$E_{3(+-)(-+)}^{\pm} \equiv \frac{1}{2\pi^2}\int D^a Y_k^{-\frac{1}{2}k,\frac{1}{2}k}Y_k^{\frac{1}{2}k,-\frac{1}{2}k}Y_a^{3\pm} = ik. \tag{A.26}$$

Vector spherical harmonics $Y_{ka}^{0\pm}$ whose Lie derivatives along the SO(2) directions are zero can be expressed as

$$Y_k^{0+} = \frac{1}{\sqrt{2}}(\sin^2\theta p_l(\theta)d\phi + \cos^2\theta q_l(\theta)d\psi); \tag{A.27}$$

$$Y_k^{0-} = \frac{1}{\sqrt{2}}(-\sin^2\theta p_l(\theta)d\phi + \cos^2\theta q_l(\theta)d\psi), \tag{A.28}$$

where $k = 2l+1$ and l is an integer. The functions $p_l(\theta)$ and $q_l(\theta)$ of degree $2l$ are related to degree $k = 2l+1$ scalar harmonics with $\mathrm{SO}(2) \times \mathrm{SO}(2)$ charges $(\pm\frac{1}{2},\pm\frac{1}{2})$. That is,

$$Y_k^{\pm\frac{1}{2},\pm\frac{1}{2}}(\theta) = e^{\pm i\phi}\sin\theta p_l(\theta); \quad Y_k^{\pm\frac{1}{2},\mp\frac{1}{2}}(\theta) = e^{\pm i\psi}\cos\theta q_l(\theta), \tag{A.29}$$

are normalized degree k spherical harmonics. Explicit series representation of these functions are

$$p_l(\theta) = \sqrt{k+1}\left(\sum_{m=0}^{l}(-)^m\binom{l}{m}\binom{l+m+1}{l+1}(\cos\theta)^{2m}\right); \tag{A.30}$$

$$q_k(\theta) = \sqrt{k+1}\left(\sum_{m=0}^{l}(-)^m\binom{l}{m}\binom{l+m+1}{l+1}(\sin\theta)^{2m}\right).$$

Finally, let us make explicit the relation between spherical harmonics and traceless symmetric tensors on R^4. There is a one to one map between scalar spherical harmonics of degree k and rank k symmetric traceless tensors. Given the spherical harmonic, one can read off the associated tensor by lifting it onto a sphere in R^4. For example, for the charged harmonics (A.29), we get

$$Y_k^{\pm\frac{1}{2},\pm\frac{1}{2}}(\theta) \to C_k^{\pm\frac{1}{2},\pm\frac{1}{2}} = (x^1 \pm ix^2)p_l(x); \tag{A.31}$$

$$p_l(x) = \sqrt{k+1}\left(\sum_{m=0}^{l}(-)^m\binom{l}{m}\binom{l+m+1}{l+1}((x^1)^2+(x^2)^2)^m(\sum_i(x^i)^2)^{l-m}\right).$$

B. Proof of addition theorem for harmonic functions on R^4

To prove the addition theorem one first writes

$$|x-y|^{-2} = \frac{1}{r^2}\sum_{n=0}^{\infty}\sum_{m\geq 0}^{n}(-1)^{n+m}\frac{n!}{m!(n-m)!}\frac{y^{2n-m}}{r^{2n-m}}(2\hat{x}\cdot\hat{y})^m, \tag{B.1}$$

where $x^i = r\hat{x}^i$ and $y^i = y\hat{y}^i$ with $(\hat{x}^i, \hat{y}^i)$ unit vectors. Collecting together terms of the same radial power and summing the finite series one finds

$$|x-y|^{-2} = \sum_{k\geq 0}\frac{y^k}{r^{2+k}}\frac{\sin((k+1)\gamma)}{\sin(\gamma)}, \tag{B.2}$$

where the angle γ is defined as $\cos\gamma = \hat{x}\cdot\hat{y}$.

Now at each degree k there is precisely one SO(3) invariant spherical harmonic and the normalized such harmonic is given by

$$Y_k^0(\gamma) = \sin((k+1)\gamma)/\sin(\gamma). \tag{B.3}$$

One can show this using spherical coordinates adapted to the SO(3) symmetry group, namely

$$ds_3^2 = d\hat{\theta}^2 + \sin^2\hat{\theta}d\Omega_2^2. \tag{B.4}$$

Then $Y_k^0(\hat{\theta})$ satisfies the degree k SO(3) invariant spherical harmonic equation

$$\left(\frac{1}{\sin^2\hat{\theta}}\partial_{\hat{\theta}}(\sin^2\hat{\theta}\partial_{\hat{\theta}}) + k(k+2)\right)Y_k^0(\hat{\theta}) = 0, \tag{B.5}$$

and is normalized as in the previous section. Therefore the addition theorem amounts to proving the following identity

$$Y_k^0(\gamma) = \alpha_k\sum_I Y_k^I(\theta_3^x)Y_k^I(\theta_3^y), \tag{B.6}$$

where $Y_k^I(\theta_3)$ are (normalized) spherical harmonics of degree k on the S^3 and $\alpha_k = 1/(k+1)$. First note that in the coordinate system (B.4) on the sphere

$$\cos\gamma = \cos\theta_x\cos\theta_y + \sin\theta_x\sin\theta_y(\cos\gamma_2), \tag{B.7}$$

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where γ_2 is the angle separating the vectors on the S^2. Thus when $\theta_y = 0$ (it lies on the "axis") $\cos\gamma = \cos\theta_x$. Since the SO(3) singlet harmonic is the only harmonic at level k which is non-vanishing on the axis (B.6) collapses to

$$Y_k^0(\gamma) = \alpha_k Y_k^0(\theta_x)Y_k^0(0),\tag{B.8}$$

which is true if $\alpha_k = 1/(k+1)$ since from (B.3) $Y_k^0(0) = (k+1)$.

Now consider rotating the axes so that θ_y is no longer zero. Then the function $Y_k^0(\gamma)$ still satisfies the covariant version of (B.5), namely

$$(\Box_x + k(k+2))Y_k^0(\gamma) = 0,\tag{B.9}$$

where $\Box_x$ is the Laplacian on the S^3 with coordinates θ_3^x. In other words, the function can always be expanded in spherical harmonics of rank k as

$$Y_k^0(\gamma) = \sum_I \alpha_k^I(\theta_3^y)Y_k^I(\theta_3^x),\tag{B.10}$$

where the coefficients are given by

$$\alpha_k^I(\theta_3^y) = \int_{S^3} d\Omega_3 Y_k^I(\theta_3^x)Y_k^0(\cos\gamma).\tag{B.11}$$

However, a generic function can be expanded in terms of spherical harmonics as

$$f(\theta_3^x) = \sum_{k,I} \beta_{kI}Y_k^I(\theta_3^x),\tag{B.12}$$

where

$$\beta_{kI} = \int_{S^3} d\Omega_3 f(\theta_3^x)Y_k^I(\theta_3^x),\tag{B.13}$$

and in particular for the SO(3) singlet coefficients

$$\beta_k = \int_{S^3} d\Omega_3 f(\theta_3^x)Y_k^0(\theta_x),\tag{B.14}$$

so that $f(\theta_x = 0) = \sum_k \beta_k(k+1)$. Then (B.11) is the SO(3) singlet coefficient in an expansion of the function $Y_k^I(\theta_3^x)$ in a series of $Y_k^I(\gamma,\cdots)$ (i.e. with respect to the rotated axis discussed earlier). One can thus read off the coefficient (B.11) as

$$\alpha_k^I(\theta_3^y) = (k+1)^{-1}Y_k^I(\theta_3(\gamma,\cdots))_{\gamma=0} = (k+1)^{-1}Y_k^I(\theta_3^y),\tag{B.15}$$

since in the limit $\gamma \to 0$ the angles $(\theta,\cdots)$ go over into $(\theta_y,\cdots)$. This completes the proof of (B.6).

C. Six dimensional field equations to quadratic order

In this appendix we summarize the computation of the relevant quadratic corrections to the six-dimensional field equations using the results of [26, 27]. Expanding the Einstein equation (3.11) up to second order in fluctuations gives

$$R_{MN}^{(1)} + R_{MN}^{(2)} = H_{MPQ}^A H_N^{A\,PQ} - 2(h^{KL} - h^{KP}h^L{}_P)H_{MKQ}^A H_{NL}^{A}{}^{Q} \tag{C.1}$$
$$+ h^{KL}h^{PQ}H_{MKP}^A H_{NLQ}^A + D_M\Phi D_N\Phi,$$
$$\equiv (E_{MN}^{(1)} + E_{MN}^{(2)}) \tag{C.2}$$

where

$$R_{MN}^{(1)} = D_K h_{MN}^K - \frac{1}{2}D_M D_N(h_L^L); \tag{C.3}$$
$$R_{MN}^{(2)} = -D_K(h_L^K h_{MN}^L) + \frac{1}{4}D_M D_N(h^{KL}h_{KL}) + \frac{1}{2}h_{MN}^K D_K(h_L^L) - h_{ML}^K h_{KN}^L;$$
$$h_{MN}^K \equiv \frac{1}{2}(D_M h_N^K + D_N h_M^K - D^K h_{MN}).$$

The quantities $(E_{MN}^{(1)}, E_{MN}^{(2)})$ are defined to be linear and quadratic in fluctuations respectively. The expansion of the scalar field and the three forms G^A (4.1) implies the following expansion for the three forms H^A up to quadratic order in fluctuations:

$$H^5 = g^o + g^5 + \Phi g^6 + \frac{1}{2}g^o\Phi^2; \tag{C.4}$$
$$H^6 = g^6 + g^o\Phi + g^5\Phi,$$

where (g^5, g^6) are the (closed) three form fluctuations given in (4.1) and g^o is the background three form.

The scalar field equation up to second order is

$$(\Box + \Box_a)\Phi \equiv E^{(1)} + E^{(2)}; \tag{C.5}$$
$$= D_K\Phi(D_L h^{KL} - \frac{1}{2}D^K(h_L^L) + h^{KL}D_K D_L\Phi + \frac{2}{3}H_{KLM}^5(H^{6KLM} - 3h_S^K H^{6SLM}),$$

where $E^{(1)}$ is the part linear in fluctuations and $E^{(2)}$ is quadratic part. Recall that $\Box$ is the d'Alambertian on AdS_3 and $\Box_a$ is the d'Alambertian on S^3.

The (anti)-self duality equation is

$$H \mp *H \pm S^{(1)} \pm S^{(2)} \equiv T^{(1)} + T^{(2)} = 0, \tag{C.6}$$

where

$$S_{KLM}^{(1)} = \frac{1}{2}h(*H)_{KLM} - 3h_{[K}^P(*H)_{LM]P}; \tag{C.7}$$
$$S_{KLM}^{(2)} = \frac{3}{2}h_P^P h_{[K}^Q(*H)_{LM]Q} - \left(\frac{1}{8}h^2 + \frac{1}{4}h^{PQ}h_{PQ}\right)(*H)_{KLM} - 3h_{[K}^P h_L^Q(*H)_{M]PQ},$$

and $(T^{(1)}, T^{(2)})$ are the parts linear and quadratic in fluctuations respectively.

We are interested in corrections to the $(s^2, \sigma^2, H_{\mu\nu}, A_\mu^\pm)$ field equations quadratic in the scalar field s^1 and the gauge field $A^\pm$. Consider first the s^2 field equations. The linearized field equation is given by a combination of the scalar field equation (C.5) and components of the anti-self-duality equation (C.6). That is,

$$\Box s_I^2 \equiv \frac{1}{12}\left((\Box + \Box_a)\Phi - E^{(1)} - \epsilon^{abc}\left(\frac{1}{2}D^\mu D^a T_{\mu bc}^{(1)} + \frac{2}{3}T_{abc}^{(1)}\right)\right)_{Y_2^I} = 0, \qquad \text{(C.8)}$$

where $A_{Y_2^I}$ denotes the projection of A onto the Y_2^I harmonic. For the quadratic corrections to this equation first define the following quantities

$$q_1 = E^{(2)}; \quad q_{2\mu a} = -\frac{1}{2}\epsilon^{abc}T_{\mu bc}^{(2)6}; \quad q_3 = \frac{1}{6}\epsilon^{abc}T_{abc}^{(2)6}, \qquad \text{(C.9)}$$

then the correction to the s_I^2 equation is given by

$$\Box s_I^2 = \frac{1}{12}((q_1) + D^\mu D^a (q_{2\mu a}) + 4(q_3))_{Y_2^I}. \qquad \text{(C.10)}$$

Now the explicit computations of [27] show that there are no such correction terms quadratic in S_i^1 and $A^{\pm\alpha}$. Therefore the linearized equation remains uncorrected to quadratic order.

Next consider the σ_I^2 equation. Here the linearized equation is a specific combination of the components of the Einstein equation (C.2) along the sphere with components of the self-duality equation. Namely

$$\Box \sigma_I^2 \equiv \frac{1}{6}\left(\frac{1}{3}(E_a^{(1)a} - R_a^{(1)a}) + \frac{1}{4}(E_{(ab)}^{(1)} - R_{(ab)}^{(1)}) - \frac{1}{4}\epsilon^{\mu\nu\rho}D_\mu D^a T_{\nu\rho a}^{(1)5} + \frac{2}{3}\epsilon^{abc}T_{abc}^{(1)5}\right)_{Y_2^I} = 0. \qquad \text{(C.11)}$$

For the quadratic corrections to this equation define

$$Q_1 = \frac{1}{3}(E_a^{(2)a} - R_a^{(2)a}); \quad Q_{2(ab)} = (E_{(ab)}^{(2)} - R_{(ab)}^{(2)}); \qquad \text{(C.12)}$$
$$Q_{3a}^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho}T_{\nu\rho a}^{(2)5}; \quad Q_4 = \frac{1}{3!}\epsilon^{abc}T_{abc}^{(2)5},$$

and again denote as $(Q)_{Y_k^I}$ the projection of Q onto Y_k^I. Then

$$\Box \sigma^2 = \frac{1}{6}(Q_1 + \frac{1}{4}D^a D^b Q_{2(ab)} - \frac{1}{2}D^\mu D^a Q_{3a\mu} + 4Q_4)_{Y_2^I}. \qquad \text{(C.13)}$$

Now the terms quadratic in the scalar fields s^1 were computed in [27]

$$(Q_1)_{Y^I} = -14 s_i^1 s_j^1 a_{Iij} + \frac{2}{3}(D_\mu s_i^1 D^\mu s_j^1 + 2 s_i^1 s_j^1) b_{Iij}; \qquad \text{(C.14)}$$
$$(D^a D^b Q_{2(ab)})_{Y^I} = 4\left(s_i^1 s_j^1 - D_\mu s_i^1 D^\mu s_j^1\right) d_{ijI};$$
$$(D^\mu D^a Q_{3a\mu})_{Y^I} = -4\left(s_i^1 s_j^1 - D_\mu s_i^1 D^\mu s_j^1\right) b_{iIj};$$
$$(Q_4)_{Y^I} = 4 s_i^1 s_j^1 a_{Iij}.$$

The relevant spherical harmonic triple overlaps are defined in appendix A. We should mention here that there are also contributions to (C.13) quadratic in the gauge field which were not explicitly computed in [27]. These are given by

$$(Q_1)_{Y^I} = -\frac{1}{8}F_{\mu\nu}(A^{+\alpha})F^{\mu\nu}(A^{-\beta})f_{I\alpha\beta} + \cdots;$$ (C.15)

$$(D^a D^b Q_{2(ab)})_{Y^I} = -\frac{5}{2}F_{\mu\nu}(A^{+\alpha})F^{\mu\nu}(A^{-\beta})f_{I\alpha\beta} + \cdots;$$

$$(D^\mu D^a Q_{3a\mu})_{Y^I} = \frac{3}{4}D_\mu\left(F^{\mu\nu}(A^{+\alpha})A_\nu^{-\beta} + F^{\mu\nu}(A^{-\beta})A_\nu^{+\alpha}\right)f_{I\alpha\beta} + \cdots.$$

The spherical harmonic triple overlap $f_{Ia\beta}$ is defined in (A.9). Terms quadratic in two $SU(2)_L$ gauge fields or two $SU(2)$ right gauge fields are projected out via the identities (A.12). The ellipses denote terms quadratic in the gauge field rather than its field strength, that is, proportional to $A_\mu^{\pm\alpha}A^{\mu\pm\beta}$. These terms cancel out when combined in (C.13) leaving only a contribution involving field strengths. The latter however vanish when one imposes the leading order field equations, and thus the combination of the corrections (C.14) and (C.15) gives the σ^2 field equation (5.8), containing only scalar field corrections.

Next consider the corrections to the Einstein equation. Recall that the three dimensional metric to quadratic order in the fields is

$$H_{\mu\nu} = h_{\mu\nu}^0 + \pi^0 g_{\mu\nu}^o - h_\mu^{\pm\alpha}h_\nu^{\pm\alpha} \equiv \hat{H}_{\mu\nu} - h_\mu^{\pm\alpha}h_\nu^{\pm\alpha}.$$ (C.16)

Then one can show that

$$(\mathcal{L}_\mathcal{E} + 2)\hat{H}_{\mu\nu} = (E_{\mu\nu}^{(2)} - R_{\mu\nu}^{(2)})_{Y_0} + (3Q_1 + 4Q_4)_{Y_0}g_{\mu\nu}^o,$$ (C.17)

where the linearized Einstein operator is defined in (5.5). The following terms which are quadratic in the scalar fields

$$(E_{\mu\nu}^{(2)} - R_{\mu\nu}^{(2)})^0 = (-2s_i^1 s_j^1 g_{\mu\nu}^o + 16D_\mu s_i^1 D_\nu s_j^1 - 6D_\rho s_i^1 D^\rho s_j^1 g_{\mu\nu}^o)\delta^{ij},$$ (C.18)

in combination with those contained in (C.14) give

$$(\mathcal{L}_\mathcal{E} + 2)\hat{H}_{\mu\nu} = 16(D_\mu s_i^1 D_\nu s_i^1 - g_{\mu\nu}^o s_i^1 s_i^1).$$ (C.19)

There are also contributions quadratic in the gauge fields to both $(\mathcal{L}_\mathcal{E} + 2)\hat{H}_{\mu\nu}$ and $(\mathcal{L}_\mathcal{E} + 2)h_\mu^{\pm\alpha}h_\nu^{\pm\alpha}$. These contributions involve both the gauge fields and their field strength, and in particular do not vanish for flat connections. This is unsurprising, since we know from general arguments that $\hat{H}_{\mu\nu}$ on its own does not transform correctly under gauge transformations. However the gauge field contributions to $(\mathcal{L}_\mathcal{E} + 2)H_{\mu\nu}$, where $H_{\mu\nu}$ is the three dimensional metric (5.9) that transforms correctly under diffeomorphisms, do vanish for flat connections, as indeed they should, and thus are zero when one imposes the leading order gauge field equations. The corrected Einstein equation is therefore that given in (5.10).

D. 3-point functions

In this appendix we discuss the supergravity computation of certain 3-point functions.

D.1 Extremal scalar three point functions

First we will consider the computation of the 3-point function between two operators of dimension 1 and one operator of dimension k. The operators of dimension 1 may be the same or different and are dual to the fields S^1; there are four such operators corresponding to the four scalar harmonics of degree 1 which are labeled by i, j. The operator $\mathcal{O}_{\Sigma_I^k}$ of dimension k is dual to the field Σ_I^k (there are $(k+1)^2$ such operators labeled by I). The $k = 2$ case is special in that the correlator is extremal [25]. As in the five dimensional case, the computation of extremal correlators is subtle. The bulk coupling vanishes but the spacetime integral diverges when $k \to 2$ in such way that the corresponding 3-point function is finite. We will take this value to be the correct extremal correlator and this will allow us to fix the coefficient of the relevant terms non-linear in momentum in the 1-point function of Σ^2.

The three dimensional field equations to quadratic order were determined in [26] and for the fields of interest and with our normalizations they read

$$(\Box - k(k-2))\Sigma_I^k = w_{Iij} S_i^1 S_j^1; \tag{D.1}$$
$$(\Box + 1)S_i^1 = w_{Iij}\Sigma_I^k S_j^1;$$
$$(\Box + 1)S_j^1 = w_{Iij}\Sigma_I^k S_i^1;$$

where

$$w_{Iij} = \frac{k^3(k+2)(k+4)(1-k/2)}{32(k+1)\sqrt{k(k-1)}} a_{Iij}. \tag{D.2}$$

Notice that this coupling vanishes in the extremal case $k = 2$.

The aim is to compute the 3-point $\langle \mathcal{O}_{\Sigma_I^k}(x_1) \mathcal{O}_{S_i^1}(x_2) \mathcal{O}_{S_j^1}(x_3) \rangle$, but we start by discussing 2-point functions. These are obtained by the first variation of the 1-point functions

$$\langle \mathcal{O}_{\Sigma_I^k}(x_1) \mathcal{O}_{\Sigma_J^k}(x_2) \rangle = -\frac{\delta \langle \mathcal{O}_{\Sigma_I^k}(x_1) \rangle}{\delta \Sigma_{J(0)}^k(x_2)} = -\left(\frac{n_1 n_5}{4\pi}\right)(2k-2)\frac{\delta \Sigma_{I(2k-2)}^k(x_1)}{\delta \Sigma_{J(0)}^k(x_2)};$$

$$\langle \mathcal{O}_{S_i^1}(x_1) \mathcal{O}_{S_j^1}(x_2) \rangle = -\frac{\delta \langle \mathcal{O}_{S_i^1}(x_1) \rangle}{\delta S_{j(0)}^1(x_2)} = -\left(\frac{n_1 n_5}{4\pi}\right)2\frac{\delta \tilde{S}_{i(0)}^1(x_1)}{\delta S_{j(0)}^1(x_2)},$$

where we used (5.22). It follows that in order to obtain these 2-point functions we need to solve (D.1) to linear order in the sources (so the r.h.s is set equal to zero) and then extract the appropriate coefficient. The details of this computation can be found in section 6.3 of [47] with the following result

$$\langle \mathcal{O}_{\Sigma_I^k}(x_1) \mathcal{O}_{\Sigma_J^k}(x_2) \rangle = \left(\frac{n_1 n_5}{4\pi}\right)\frac{(2k-2)\Gamma(k)}{\pi\Gamma(k-1)}\left(\frac{1}{x^{2k}}\right)_R \delta_{IJ}, \qquad k \neq 1;$$

$$\langle \mathcal{O}_{S_i^1}(x_1) \mathcal{O}_{S_j^1}(x_2) \rangle = \left(\frac{n_1 n_5}{4\pi}\right)\frac{2}{\pi}\left(\frac{1}{x^2}\right)_R \delta_{ij}, \tag{D.3}$$

where the subscript R indicates that these are renormalized correlators.

We now discuss the 3-point function with $k \neq 2$. We can can obtain the 3-point function by the second variation of the 1-point function of O_{Σ^k}:

$$
\begin{aligned}
\langle \mathcal{O}_{\Sigma_i^k}(x_1) \mathcal{O}_{S_i^1}(x_2) O_{S_j^1}(x_3) \rangle &= \frac{\delta^2 \langle \mathcal{O}_{\Sigma_I^k}(x_1) \rangle}{\delta S_{i(0)}^1(x_2) \delta S_{j(0)}^1(x_3)} \\
&= \left(\frac{n_1 n_5}{4\pi} \right) (2k - 2) \frac{\delta^2 \Sigma_{I(2k-2)}^k(x_1)}{\delta S_{i(0)}^1(x_2) \delta S_{j(0)}^1(x_3)}
\end{aligned} \tag{D.4}
$$

It follows that we need to solve (D.1) to quadratic order in the sources and then extract the coefficient of order z^k. The steps involved in this computation are spelled out in section 5.9 of [21]. For the case at hand, the result is[11]

$$
\langle \mathcal{O}_{\Sigma_I^k}(x_1) \mathcal{O}_{S_i^1}(x_2) O_{S_j^1}(x_3) \rangle = - \left(\frac{n_1 n_5}{4\pi} \right) w_{Iij} \frac{2\Gamma(k)}{\pi^3 \Gamma(k-1)} I_k(x_1, x_2, x_3) \tag{D.5}
$$

where

$$
I_k(x_1, x_2, x_3) = \int \frac{d^2 x \, dz}{z^3} \left(\frac{z}{z^2 + (\vec{x} - \vec{x}_1)^2} \right)^k \left(\frac{z}{z^2 + (\vec{x} - \vec{x}_2)^2} \right) \left(\frac{z}{z^2 + (\vec{x} - \vec{x}_3)^2} \right). \tag{D.6}
$$

This integral was computed in [45] with answer

$$
I_k(x_1, x_2, x_3) = \frac{\pi \Gamma(1 - k/2)(\Gamma(k/2))^3}{2\Gamma(k)} \frac{1}{|\vec{x}_1 - \vec{x}_2|^k |\vec{x}_1 - \vec{x}_3|^k |\vec{x}_2 - \vec{x}_3|^{2-k}}. \tag{D.7}
$$

Notice that this integral diverges in the extremal case $k \to 2$.

The final answer for the correlator is thus

$$
\langle \mathcal{O}_{\Sigma_I^k}(x_1) \mathcal{O}_{S_i^1}(x_2) O_{S_j^1}(x_3) \rangle = \frac{C_{Iij}^k}{|\vec{x}_1 - \vec{x}_2|^k |\vec{x}_1 - \vec{x}_3|^k |\vec{x}_2 - \vec{x}_3|^{2-k}} \tag{D.8}
$$

where

$$
C_{Iij}^k = - \left(\frac{n_1 n_5}{4\pi} \right) \frac{k^3 (k+2)(k+4) \Gamma(k/2)^3 \Gamma(2 - k/2)}{32\pi^2 (k+1) \Gamma(k-1) \sqrt{k(k-1)}} a_{Iij}. \tag{D.9}
$$

This coefficient has a smooth limit as $k \to 2$; the zero in w_{Iij} cancels against the divergence in I_2, and we get

$$
C_{Iij}^2 = - \left(\frac{n_1 n_5}{4\pi} \right) \frac{1}{\sqrt{2}\pi^2} a_{Iij}. \tag{D.10}
$$

We will take this to be the correct extremal 3-point function, i.e.,

$$
\langle \mathcal{O}_{\Sigma_I^2}(x_1) \mathcal{O}_{S_i^1}(x_2) O_{S_j^1}(x_3) \rangle = \frac{C_{Iij}^2}{|\vec{x}_1 - \vec{x}_2|^2 |\vec{x}_1 - \vec{x}_3|^2}, \tag{D.11}
$$

and use it to deduce the non-linear coupling in the 1-point function of $\langle \mathcal{O}_{\Sigma_I^2} \rangle$. As discussed in [20], the form of the 1-point function is uniquely fixed by general arguments to be

$$
\langle \mathcal{O}_{\Sigma_I^2} \rangle = \left(\frac{n_1 n_5}{4\pi} \right) \left(\pi_{(2)}^{\Sigma_I^2} + A_{Iij} \pi_{(1)}^{S_i^1} \pi_{(1)}^{S_j^1} \right) \tag{D.12}
$$

---

[11]The normalization of the bulk-to-boundary propagator in (5.52) when $\Delta = 1$ is $C_1 = 1/\pi$.

The numerical coefficient A_{Iij} should be determined by doing holographic renormalization in 6 (rather than 3) dimensions. We will fix it, however, such that the the extremal correlator is correctly computed directly at $k = 2$ (rather than obtained as a limit from $k \neq 2$). Since $w_{Iij}(k = 2) = 0$ the only contribution comes from the terms non-linear in momenta

$$\langle \mathcal{O}_{\Sigma_I^k}(x_1) \mathcal{O}_{S_i^1}(x_2) \mathcal{O}_{S_j^1}(x_3) \rangle = \left(\frac{n_1 n_5}{4\pi}\right) 2 A_{Iij} \left(\frac{\delta \pi_{(1)}^{S_i^1}(x_1)}{S_{i(0)}^1(x_2)}\right) \left(\frac{\delta \pi_{(1)}^{S_j^1}(x_1)}{S_{j(0)}^1(x_3)}\right);$$

$$= \left(\frac{n_1 n_5}{4\pi}\right) A_{Iij} \frac{8}{\pi^2} \frac{1}{|\vec{x}_1 - \vec{x}_2|^2 |\vec{x}_1 - \vec{x}_3|^2} \quad \text{(D.13)}$$

By comparing with (D.11) we find

$$A_{Iij} = -\frac{1}{4\sqrt{2}} a_{Iij}. \quad \text{(D.14)}$$

D.2 Non-extremal scalar three point functions

We will also need other three-point functions for scalars due to chiral primary operators. The relevant cubic couplings in three dimensions were also computed in [36, 26] and are given by

$$-\frac{n_1 n_5}{4\pi} \int d^3x \sqrt{-G} (T_{123} S^1 S^2 \Sigma^3 + U_{123} \Sigma^1 \Sigma^2 \Sigma^3); \quad \text{(D.15)}$$

$$\equiv -\frac{n_1 n_5}{16\pi} \int d^3x \sqrt{-G} V_{123} \left(\frac{S^1 S^2 \Sigma^3}{\sqrt{(k_1+1)(k_2+1)}} + \frac{(k_1^2 + k_2^2 + k_3^2 - 2)}{(k_1 + 1)(k_2 + 1)} \frac{\Sigma^1 \Sigma^2 \Sigma^3}{6\sqrt{(k_1-1)(k_2-1)}} \right),$$

$$V_{123} = \frac{\Sigma(\Sigma + 2)(\Sigma - 2) \alpha_1 \alpha_2 \alpha_3 a_{123}}{(k_3 + 1)\sqrt{k_1 k_2 k_3 (k_3 - 1)}}$$

where k_a denotes the dimension of the operator dual to the field Ψ^a, $\Sigma = k_1 + k_2 + k_3$, $\alpha_1 = \frac{1}{2}(k_2 + k_3 - k_1)$ etc and a_{123} is shorthand for the spherical harmonic overlap. It is straightforward to follow the same steps as before to compute the associated three point functions:

$$\langle \mathcal{O}_{S^1}(x_1) \mathcal{O}_{S^2}(x_2) \mathcal{O}_{\Sigma^3}(x_3) \rangle = \frac{3N}{4\pi^3} \frac{W_{123} T_{123}}{|\vec{x}_1 - \vec{x}_2|^{2\alpha_3} |\vec{x}_1 - \vec{x}_3|^{2\alpha_2} |\vec{x}_2 - \vec{x}_3|^{2\alpha_1}}; \quad \text{(D.16)}$$

$$\langle \mathcal{O}_{\Sigma^1}(x_1) \mathcal{O}_{\Sigma^2}(x_2) \mathcal{O}_{\Sigma^3}(x_3) \rangle = \frac{3N}{4\pi^3} \frac{W_{123} U_{123}}{|\vec{x}_1 - \vec{x}_2|^{2\alpha_3} |\vec{x}_1 - \vec{x}_3|^{2\alpha_2} |\vec{x}_2 - \vec{x}_3|^{2\alpha_1}};$$

$$W_{123} = -\frac{\Gamma(\alpha_1 + 1)\Gamma(\alpha_2 + 1)\Gamma(\alpha_3 + 1)\Gamma(\frac{1}{2}(\Sigma - 2))}{\Gamma(k_1 - 1)\Gamma(k_2 - 1)\Gamma(k_3 - 1)}.$$

We will be interested in the case where $(S^1, S^2, \Sigma^1, \Sigma^2)$ have dimension k and (S^2, Σ^2) are chiral primary with (S^1, Σ^1) anti-chiral primary. Then charge conservation implies that the correlators are only non-zero when Σ^3 is neutral. In the case where $\mathcal{O}_{\Sigma^3}$ has dimension two the explicit results for the correlators using the spherical harmonic overlap of (A.24)

are

$$\langle(\mathcal{O}_{S_k^p})^\dagger(x_1)\mathcal{O}_{S_k^p}(x_2)\mathcal{O}_{\Sigma_0^2}(x_3)\rangle = \frac{N\sqrt{3}}{2\sqrt{2}\pi^3}\frac{k^3}{|\vec{x}_1-\vec{x}_2|^{2(k-1)}|\vec{x}_1-\vec{x}_3|^2|\vec{x}_2-\vec{x}_3|^2}; \quad (D.17)$$

$$\langle(\mathcal{O}_{\Sigma_k^p})^\dagger(x_1)\mathcal{O}_{\Sigma_k^p}(x_2)\mathcal{O}_{\Sigma_0^2}(x_3)\rangle = \frac{N\sqrt{3}}{2\sqrt{2}\pi^3(k+2)^3}\frac{k(k-1)(k^4-1)}{|\vec{x}_1-\vec{x}_2|^{2(k-1)}|\vec{x}_1-\vec{x}_3|^2|\vec{x}_2-\vec{x}_3|^2}.$$

It will be useful to define normalized three point functions as

$$\langle(\mathcal{O}_{S_k^p})^\dagger\mathcal{O}_{\Sigma_0^2}(x)\mathcal{O}_{S_k^p}\rangle \equiv \frac{\langle(\mathcal{O}_{S_k^p})^\dagger(\infty)\mathcal{O}_{\Sigma_0^2}(x)\mathcal{O}_{S_k^p}(0)\rangle}{\langle(\mathcal{O}_{S_k^p})^\dagger(\infty)\mathcal{O}_{S_k^p}(0)\rangle} = \frac{\sqrt{3}k^3}{\sqrt{2}\pi(k-1)^2}\frac{1}{|\vec{x}|^2}. \quad (D.18)$$

$$\langle(\mathcal{O}_{\Sigma_k^p})^\dagger\mathcal{O}_{\Sigma_0^2}(x)\mathcal{O}_{\Sigma_k^p}\rangle \equiv \frac{\langle(\mathcal{O}_{\Sigma_k^p})^\dagger(\infty)\mathcal{O}_{\Sigma_0^2}(x)\mathcal{O}_{\Sigma_k^p}(0)\rangle}{\langle(\mathcal{O}_{\Sigma_k^p})^\dagger(\infty)\mathcal{O}_{\Sigma_k^p}(0)\rangle};$$

$$= \frac{\sqrt{3}k(k+1)(k^2+1)}{\sqrt{2}\pi(k+2)^2}\frac{1}{|\vec{x}|^2}.$$

(Implicitly we assume here that $k \neq 1$.) Note that for $k \gg 1$ these expressions both tend to the same limit, $\sqrt{3}k/\sqrt{2}\pi|\vec{x}|^2$.

D.3 Two scalars and R symmetry current

Finally we will need three point functions between two scalars (of the same mass) and the R symmetry current. The relevant cubic couplings were again given in [26]:

$$-\frac{n_1 n_5}{8\pi}\int d^3x\sqrt{-G}A_\mu^{\pm\alpha}(S_I^k D_\mu S_J^k + \Sigma_I^k D_\mu \Sigma_J^k)E_{\alpha IJ}^\pm, \quad (D.19)$$

where the triple overlap is defined in (A.11). To compute the corresponding three point functions one again follows the steps given in [21]. This results in

$$\langle\mathcal{O}_{S_I^k}(x_1)J^{\pm\alpha}(x)\mathcal{O}_{S_J^k}(x_2)\rangle = \langle\mathcal{O}_{\Sigma_I^k}(x_1)J^{\pm\alpha}(x)\mathcal{O}_{\Sigma_J^k}(x_2)\rangle = \mp i\frac{N}{8\pi}E_{\alpha IJ}^\pm I_\mp(x,x_1,x_2), \quad (D.20)$$

where the AdS integral

$$I_\mp(x,x_1,x_2) = \int\frac{d^3z}{z^3}K_k(z,\vec{x}_1)D^\mu K_k(z,\vec{x}_2)\mathcal{G}_{\mu\mp}(z,\vec{x}) = \frac{(k-1)^2}{\pi^2}\frac{Z_\mp}{|\vec{x}_1-\vec{x}_2|^{2k}}, \quad (D.21)$$

was computed in [45]. In this integral $K_k(z,\vec{x})$ and $\mathcal{G}_{\mu\mp}(z,\vec{x})$ are the standard AdS scalar and vector bulk to boundary propagators respectively and

$$Z_+ = \frac{1}{(w_1-w)} - \frac{1}{(w_2-w)}; \quad Z_- = \frac{1}{(\bar{w}_1-\bar{w})} - \frac{1}{(\bar{w}_2-\bar{w})}. \quad (D.22)$$

Here we have implicitly switched to Euclidean signature, $t = i\tau$, and introduced complex boundary coordinates $w = y + i\tau$.

In deriving this result we use the standard vector propagator, that following from the field equation $D_\mu F^{\mu\nu} = 0$, although the (linearized) vector equation here is Chern-Simons, $F_{\mu\nu} = 0$. Whilst this step should be justified more rigorously, it can be justified

a posteriori by the fact that the three point functions thus obtained are of the standard form for a two dimensional CFT. To see this, consider the case where the scalar operators are chiral primary. Using the specific values for the spherical harmonic overlaps (A.25) in (D.20) gives

$$\langle(\mathcal{O}_{S^k})^\dagger(x_1)J^{+3}(w)\mathcal{O}_{S^k}(x_2)\rangle = \frac{N}{8\pi^3}k(k-1)^2\left(\frac{1}{(w_1-w)}-\frac{1}{(w_2-w)}\right); \qquad \text{(D.23)}$$

$$= \langle(\mathcal{O}_{S^k})^\dagger(x_1)\mathcal{O}_{S^k}(x_2)\rangle\frac{k}{4\pi}\left(\frac{1}{(w_1-w)}-\frac{1}{(w_2-w)}\right),$$

with the latter being the canonical form for the CFT three point function between the (holomorphic) R current and operators charged under it. An analogous formula holds for the anti-holomorphic current, $J^{-3}(\bar{w})$ and for the correlators involving scalar operators dual to Σ^k. Again it is useful to define normalized three point functions such that

$$\langle(\mathcal{O}_{S^k})^\dagger J^{+3}(w)\mathcal{O}_{S^k}\rangle \equiv \frac{\langle(\mathcal{O}_{S^k})^\dagger(\infty)J^{+3}(w)\mathcal{O}_{S^k}(0)\rangle}{\langle(\mathcal{O}_{S^k})^\dagger(\infty)\mathcal{O}_{S^k}(0)\rangle} = \frac{k}{4\pi w}; \qquad \text{(D.24)}$$

$$\langle(\mathcal{O}_{\Sigma^k})^\dagger J^{+3}(w)\mathcal{O}_{\Sigma^k}\rangle \equiv \frac{\langle(\mathcal{O}_{\Sigma^k})^\dagger(\infty)J^{+3}(w)\mathcal{O}_{\Sigma^k}(0)\rangle}{\langle(\mathcal{O}_{\Sigma^k})^\dagger(\infty)\mathcal{O}_{\Sigma^k}(0)\rangle} = \frac{k}{4\pi w},$$

with analogous formulae holding for the anti-holomorphic currents. The corresponding normalized three point functions for the spectrally flowed operators in the R sector are then

$$\langle(\mathcal{O}_{S^k})_R^\dagger J^{+3}(w)(\mathcal{O}_{S^k})_R\rangle \equiv \frac{\langle(\mathcal{O}_{S^k})_R^\dagger(\infty)J^{+3}(w)(\mathcal{O}_{S^k})_R(0)\rangle}{\langle(\mathcal{O}_{S^k})_R^\dagger(\infty)(\mathcal{O}_{S^k})_R(0)\rangle} = \frac{k-N}{4\pi w}; \qquad \text{(D.25)}$$

$$\langle(\mathcal{O}_{\Sigma^k})_R^\dagger J^{+3}(w)(\mathcal{O}_{\Sigma^k})_R\rangle \equiv \frac{\langle(\mathcal{O}_{\Sigma^k})_R^\dagger(\infty)J^{+3}(w)(\mathcal{O}_{\Sigma^k})_R(0)\rangle}{\langle(\mathcal{O}_{\Sigma^k})^\dagger(\infty)\mathcal{O}_{\Sigma^k}(0)\rangle} = \frac{k-N}{4\pi w},$$

where $\mathcal{O}_R$ denotes the spectral flowed operator. Again corresponding formulae hold for the anti-holomorphic currents.

E. Holographic 1-point functions

In this appendix we derive the 1-point function for the stress energy tensor and the operators dual to S_i^1. We omit the details of this computation since the analysis is very similar to the Coulomb branch analysis in [41, 42]. The asymptotic analysis of this system is also presented (in a different coordinate system) in [46] and the form of the counterterm was obtained in [47].

The relevant action is given in (5.15), retaining only the graviton and scalar fields S_i^1, and the most general asymptotic solution with Dirichlet boundary conditions is given by

the expansion in (5.21) with coefficients given by

$$
\begin{aligned}
\text{Tr } g_{(2)} &= -\frac{1}{2}R - \frac{1}{2}\left(2(S^1_{i(0)})^2 + (\tilde{S}^1_{i(0)})^2\right) \\
D^v g_{(2)uv} &= -D_u\left(\frac{1}{2}R + \frac{1}{4}\left((\tilde{S}^1_{(0)i})^2 + 4(S^1_{i(0)})^2 - 2S^1_{i(0)}\tilde{S}^1_{i(0)}\right)\right) - S^1_{i(0)}D_u\tilde{S}^1_{i(0)} \\
h_{(2)uv} &= -\frac{1}{2}S^1_{i(0)}\tilde{S}^1_{i(0)}g_{(0)uv} \\
\tilde{h}_{(2)uv} &= -\frac{1}{4}(S^1_{i(0)})^2 g_{(0)uv}
\end{aligned}
\tag{E.1}
$$

The traceless transverse part of $g_{(2)}$ and $\tilde{S}^1_{i(0)}$ (as well as the sources $g_{(0)uv}$ and $S^1_{i(0)}$) are unconstrained. We will soon see that these coefficients are related to the 1-point functions.

The counterterms needed to render the on-shell action finite are

$$
S_{ct} = \frac{n_1 n_5}{4\pi}\int_{z=\epsilon} d^2x\sqrt{-\gamma}\left(2 - \log\epsilon^2\frac{1}{2}R + \frac{1}{2}(S^1_i)^2\left(1 + \frac{2}{\log\epsilon^2}\right)\right)
\tag{E.2}
$$

so the on-shell renormalized action consists of (5.15), the Gibbons-Hawking term and these counterterms (along with additional counterterms for the gauge fields, discussed in the main text). The logarithmic terms determine the holographic conformal anomalies [48].

The renormalized 1-point functions are [12]

$$
\langle\mathcal{O}_{S^1_i}\rangle = \frac{n_1 n_5}{4\pi}(2\tilde{S}^1_{i(0)});
\tag{E.3}
$$

$$
\langle T_{uv}\rangle = \frac{n_1 n_5}{2\pi}\left(g_{(2)uv} + \frac{1}{2}Rg_{(0)uv} \right.
$$
$$
\left. + \frac{1}{4}\left((\tilde{S}^1_{i(0)})^2 - 2\tilde{S}^1_{i(0)}S^1_{i(0)} + 4(S^1_{i(0)})^2\right)g_{(0)uv}\right).
$$

Using the asymptotic solution one may verify that these expressions satisfy the correct Ward identities

$$
\langle T^u_u\rangle = -S^1_{i(0)}\langle\mathcal{O}_{S^1_i}\rangle + \mathcal{A}
\tag{E.4}
$$

$$
D^v\langle T_{uv}\rangle = -\langle\mathcal{O}_{S^1_i}\rangle D_u S^1_{i(0)}.
\tag{E.5}
$$

The first term on the r.h.s. is the standard term due to the coupling of the source $S^1_{i(0)}$ to an operator of dimension one. The conformal anomaly $\mathcal{A}$ is given by

$$
\mathcal{A} = \frac{c}{24\pi}R + \frac{n_1 n_5}{2\pi}(S^1_{i(0)})^2 ; \qquad c = 6n_1 n_5
\tag{E.6}
$$

The first term is the standard gravitational conformal anomaly and the second the conformal anomaly induced by the short distance singularities in the 2-point function of $\mathcal{O}_{S^1_i}$ [49].

F. Three point functions from the orbifold CFT

In this appendix we discuss the relationship between three point functions computed in the CFT on the symmetric product $S^N(T^4)$ with those in supergravity. The chiral primary

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[12]In comparing with [47] one should note the factor of 2 difference in the source.

operators are summarized in (8.1); their detailed construction is not important here, but note that they are S_N invariant and orthonormal. The operators (8.1) manifestly have the correct dimensions and charges to correspond to the fields $S_k^{(r)I}$ and Σ_k^I in supergravity. Moreover, as discussed in section 8 the most natural correspondence seems to be that given in (8.4) although this choice is not unique.

Extremal three point functions of these operators have the following structure as $N \to \infty$ [39]

$$\left\langle \mathcal{O}_{n+k-1}^{(0,0)\dagger}(\infty)\mathcal{O}_k^{(0,0)}(1)\mathcal{O}_n^{(0,0)}(0)\right\rangle = \frac{1}{\sqrt{N}}\left((n+k-1)nk\right)^{1/2}; \tag{F.1}$$

$$\left\langle \mathcal{O}_{n+k-1}^{(i)\dagger}(\infty)\mathcal{O}_k^{(0,0)}(1)\mathcal{O}_n^{(j)}(0)\right\rangle = \frac{1}{\sqrt{N}}\left((n+k-1)nk\right)^{1/2}\delta^{ij};$$

$$\left\langle \mathcal{O}_{n+k-1}^{(2,2)\dagger}(\infty)\mathcal{O}_k^{(0,0)}(1)\mathcal{O}_n^{(2,2)}(0)\right\rangle = \frac{1}{\sqrt{N}}\left((n+k-1)nk\right)^{1/2};$$

$$\left\langle \mathcal{O}_{n+k-3}^{(2,2)\dagger}(\infty)\mathcal{O}_k^{(0,0)}(1)\mathcal{O}_n^{(0,0)}(0)\right\rangle = \frac{2}{\sqrt{N}}\left((n+k-3)nk\right)^{1/2};$$

$$\left\langle \mathcal{O}_{n+k-1}^{(2,2)\dagger}(\infty)\mathcal{O}_k^{(i)}(1)\mathcal{O}_n^{(j)}(0)\right\rangle = -\frac{1}{\sqrt{N}}\left((n+k-1)nk\right)^{1/2}\omega^i * \omega^j;$$

$$\left\langle \mathcal{O}_{n+k+1}^{(2,2)\dagger}(\infty)\mathcal{O}_k^{(2,2)}(1)\mathcal{O}_n^{(2,2)}(0)\right\rangle = 0.$$

(Here we use $\omega_{a\bar{a}}^i$ as a basis for $H^{(1,1)}(T^4)$).

The cubic couplings between scalars in supergravity were determined in [36, 26]. From (D.15) one sees that the couplings $\Sigma\Sigma\Sigma$ and ΣSS are generically non-zero whereas the couplings SSS and $S\Sigma\Sigma$ are always zero. This implies that the corresponding extremal three point functions between chiral primaries determined in supergravity have the following structures

$$\left\langle \mathcal{O}_{\Sigma_\Delta^p}^\dagger \mathcal{O}_{\Sigma_{\Delta_1}^p} \mathcal{O}_{\Sigma_{\Delta_2}^p}\right\rangle \neq 0; \quad \left\langle \mathcal{O}_{\Sigma_\Delta^p}^\dagger \mathcal{O}_{S_{\Delta_1}^p} \mathcal{O}_{S_{\Delta_2}^p}\right\rangle \neq 0; \quad \left\langle \mathcal{O}_{S_\Delta^p}^\dagger \mathcal{O}_{\Sigma_{\Delta_1}^p} \mathcal{O}_{S_{\Delta_2}^p}\right\rangle \neq 0; \tag{F.2}$$

$$\left\langle \mathcal{O}_{S_\Delta^p}^\dagger \mathcal{O}_{S_{\Delta_1}^p} \mathcal{O}_{S_{\Delta_2}^p}\right\rangle = 0; \quad \left\langle \mathcal{O}_{\Sigma_\Delta^p}^\dagger \mathcal{O}_{\Sigma_{\Delta_1}^p} \mathcal{O}_{S_{\Delta_2}^p}\right\rangle = 0; \quad \left\langle \mathcal{O}_{S_\Delta^p}^\dagger \mathcal{O}_{\Sigma_{\Delta_1}^p} \mathcal{O}_{\Sigma_{\Delta_2}^p}\right\rangle = 0,$$

where $\Delta = \Delta_1 + \Delta_2$. Note that such correlators would be determined in supergravity either by a careful limiting procedure of non-extremal correlators (which uses directly the cubic couplings mentioned above) or by reducing the six-dimensional action including all boundary terms. In the latter case given that there are no bulk couplings SSS and $S\Sigma\Sigma$ it seems that there would be no boundary couplings between such fields, and hence no non-zero extremal correlators.

The correlators (F.1) and (F.2) clearly disagree if one makes the identification proposed in (8.4). Given that this identification was not unique, one might wonder whether there is a different linear map between supergravity and orbifold CFT operators such that the correlators agree. Whilst we have not proved in full generality that this is impossible, the following argument suggests that it is unlikely. Let $\mathcal{O}_1^a = (\mathcal{O}_2^{(0,0)}, \mathcal{O}_1^{(i=1)})$ denote two of the dimension one CFT operators and $\mathcal{O}_2^\alpha = (\mathcal{O}_3^{(0,0)}, \mathcal{O}_2^{(i=1)}, \mathcal{O}_1^{(2,2)})$ denote three of the dimension two CFT operators. Let $\hat{\mathcal{O}}_1^a = \mathcal{O}_{S_1^a}$ denote two dimension one operators dual to sugra scalar fields and $\hat{\mathcal{O}}_2^\alpha = (\mathcal{O}_{S_2^a}, \mathcal{O}_{\Sigma_2})$ denote three of the dimension two operators

dual to sugra fields. Next write the fusion coefficients in the corresponding extremal three point functions in the orbifold CFT and supergravity as $C_{\alpha ab}$ and $\hat{C}_{\alpha ab}$ respectively. Since these are symmetric on the last two indices, rewrite them as (square) matrices $D_{\alpha\beta}$ and $\hat{D}_{\alpha\beta}$. Now the key point is that (F.1) and (F.2) imply that $D_{\alpha\beta}$ has non-zero determinant, but $\hat{D}_{\alpha\beta}$ has zero determinant. Any linear maps between $\mathcal{O}_1^a$ and $\hat{\mathcal{O}}_1^a$, and between $\mathcal{O}_2^\alpha$ and $\hat{\mathcal{O}}_2^\alpha$ which preserve the two point functions will not map $D_{\alpha\beta}$ to a zero determinant matrix and therefore one cannot get agreement between (F.1) and (F.2) by making a different identification between operators.

Whilst we have not extended this argument to higher dimension operators, it seems more likely that the supergravity and orbifold CFT correlators disagree because of renormalization; there is no known non-renormalization theorem for these correlators.

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